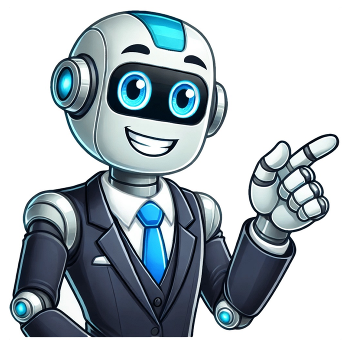

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The license may not give you all of the permissions necessary for your intended use. For example, other rights such as publicity, privacy, or moral rights may limit how you use the material. Let us consider the following equations: this equation involves sums of real expressions and multiplications by real numbers this equation involves sums of 2-d vectors and multiplications by real numbers this equation involves sums of 2 by 2 matrices and multiplications by real numbers this equation involves sums of polynommials and multiplications by real numbers. It is obvious that if the set of real numbers in equation (1), the set of 2-d vectors used in equation (2), the set of the 2 by 2 matrices used in equation (3) and the set of polynomial used in equation (4) obey some common laws of addition and multiplication by real numbers, we may be able to solve all these 4 equations, and many other more complicated questions, using the same algorithm based on the properties (or laws) of addition and multiplication by real numbers. Classifying sets by their properties helps in solving problems involving different king of mathematical objects such as matrices, polynomials, 2-d vectors, 3-d vectors, n-d vectors, planes in geometry, functions,...and developing ways and methods to solve different problems using the same algorithms. Definition of a Vector Space In what follows, vector spaces (1 , 2) are in capital letters and their elements (called vectors) are in bold lower case letters. A nonempty set V whose vectors (or elements) may be combined using the operations of addition (+) and multiplication · by a scalar is called a vector space if the conditions in A and B below are satisfied: Note An element or object of a vector space is called vector. A) the addition of any two vectors of V and the multiplication of any vectors of V by a scalar produce an element that belongs to V . Let u and v be any two elelments of the set V and r any real number. \(\) \(\) \(\) \(\) 1) \(\textbf{u} + \textbf{v} = \textbf{w}\) , \(\textbf{w} \) is an element of the set \(V\) ; we say the set \(V\) is closed under vector addition 2) \(r \cdot \textbf{u} = \textbf{z} \) , \(\textbf{z} \) is an element of the set \(V\) we say the set \(V\) is closed under scalar multiplication B) For any vectors \(\textbf{u}, \textbf{v}, \textbf{w} \) in \(V\) and any real numbers \(r \) and \(s \), the two operations described above must obey the following rules : 3) Commutatitivity of vector addition : \(\textbf{u} + \textbf{v} = \textbf{v} + \textbf{u} \) 4) Associativity of vector addition : \((\textbf{u} + \textbf{v}) +\textbf{w} = \textbf{v} + (\textbf{u} + \textbf{w}) \) 5) Associativity of multiplication: \(r \cdot (s \cdot \textbf{u}) = (r \cdot s) \cdot \textbf{u} \) 6) A zero vector \(\textbf{0} \) exists in \(\textbf{v}\) and is such that for any element \(\textbf{u}\) in the set \(\textbf{v}\), we have: \(\textbf{u} + \textbf{0} = \textbf{u} \) 7) For each vector \(\textbf{u}\) in \(V\) there exists a vector \(- \textbf{u} \) in \(V\), called the negative of \(\textbf{u}\), such that: \(\textbf{u} + (- \textbf{u}) = \textbf{0} \) 8) Distributivity of Addition of Vectors: \(r \cdot (\textbf{u} + \textbf{v}) = r \cdot \textbf{u} + r \cdot \textbf{v} \) 9) Distributivity of Addition of Real Numbers: \((r + s) \cdot \textbf{u} = r \cdot \textbf{u} + s \cdot \textbf{u} \) 10) For any element \(\textbf{u}\) in \(V\) we have: \(1 \cdot \textbf{u} = \textbf{u} \) NOTES 1) Although the element of a vector space is called vector, a vector space may be a set of matrices, functions, solutions to differential equations, 3-d vectors, ...,They do not have to be VECTORS of n dimensions such as 2 or 3 dimensional vectors used in physics. Examples of Vector Spaces Example 1 The following are examples of vector spaces: The set of all real number \(\mathbb{R} \) associated with the addition and scalar multiplication of real numbers. The set of all the complex numbers \(\mathbb{C} \) associated with the addition and scalar multiplication of complex numbers. The set of all polynomials \(R_n(x) \) with real coefficients associated with the addition and scalar multiplication of polynomials. The set of all vectors of dimension \(n \) written as \(\mathbb{R}^n \) associated with the addition and scalar multiplication as defined for 3-d and 2-d vectors for example. The set of all matrices of dimension \(m \times n \) associated with the addition and scalar multiplication as defined for matrices. The set of all functions \(\textbf{f} \) satisfying the differential equation \(\textbf{f} = \textbf{f '} \) Example 2 Proove that the set of all 2 by 2 matrices associated with the matrix addition and the scalar multiplication of matrices is a vector space. Solution to Example 2 Let \(V\) be the set of all 2 by 2 matrices. 1) Addition of matrices gives \(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} \) Adding any 2 by 2 matrices gives a 2 by 2 matrix and therefore the result of the addition belongs to \(V\). 2) Scalar multiplication of matrices gives gives \(r \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} r a & r b \\ r c & r d \end{bmatrix} \) Multiply any 2 by 2 matrix by a scalar and the result is a 2 by 2 matrix is an element of \(V\). 3) Commutativity \(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \\\\ = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} \\\\ = \begin{bmatrix} a'+a & b'+b \\ c'+c & d'+d \end{bmatrix} \\\\ = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) 4) Associativity of vector addition \(\left (\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \right) + \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix} \\ = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} + \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix} \\\\ = \begin{bmatrix} (a+a')+a'' & (b+b')+b'' \\ (c+c') + c''& (d+d')+d'' \end{bmatrix} \\\\ = \begin{bmatrix} a+(a'+a'') & b+(b'+b'') \\ c+(c' + c'')& d+(d'+d'') \end{bmatrix} \\\\ = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \left(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} + \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix} \right) \) 5) Associativity of multiplication \(r \left(s \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = r \left(\begin{bmatrix} s a & s b \\ s c & s d \end{bmatrix} \right) \\\\ = \begin{bmatrix} r s a & r s b \\ r s c & r s d \end{bmatrix} \\\\ = \begin{bmatrix} (r s) a & (r s) b \\ (r s) c & (r s) d \end{bmatrix} \\\\ = (r s) \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) 6) Zero vector \(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\\\ = \begin{bmatrix} a+0 & b+0 \\ c+0 & d+0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) 7) Negative vector \(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} - a & - b \\ - c & - d \end{bmatrix} \\\\ = \begin{bmatrix} a+(-a) & b+(-b) \\ c+(-c) & d+(-d) \end{bmatrix} \\\\ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) 8) Distributivity of sums of matrices: \(r \left (\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \right) \\\\ = \begin{bmatrix} r(a+a') & r(b+b') \\ r(c+c') & r(d+d') \end{bmatrix} \\\\ = \begin{bmatrix} r a+ r a' & r b+ r b \\ r c+r c' & r d+ r d \end{bmatrix} \\\\ = r \left (\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) + r \left(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \right) \) 9) Distributivity of sums of real numbers: \((r + s) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (r + s) a & (r + s) b \\ (r + s) c & (r + s) d \end{bmatrix} \\\\= \begin{bmatrix} r a + s a & r b + s b \\ r c + s c & r d + s d \end{bmatrix} \\\\= \begin{bmatrix} r a & r b \\ r c & r d \end{bmatrix} + \begin{bmatrix} s a & s b \\ s c & s d \end{bmatrix} \\\\= r \begin{bmatrix} a & b \\ c & d \end{bmatrix} + s \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) 10) Multiplication by 1. \(1 \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 a & 1 b \\ 1 c & 1 d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) Example 3 Show that the set of all real functions continuous on \((-\infty,\infty) \) associated with the addition of functions and the multiplication of matrices by a scalar form a vector space. Solution to Example 3 From calculus, we know if \(\textbf{f} \) and \(\textbf{g} \) are real continuous functions on \((-\infty,\infty) \) and \(r \) is a real number then \((\textbf{f} + \textbf{g})(x) = \textbf{f}(x) + \textbf{g}(x) \) is also continuous on \((-\infty,\infty) \) and \(r \textbf{f}(x) \) is also continuous on \((-\infty,\infty) \) Hence the set of functions continuous on \((-\infty,\infty) \) is closed under addition and scalar multiplication (the first two conditions above). The remaining 8 rules are automatically satisfied since the functions are real functions. Example 4 Show that the set of all real polynomials with a degree \(n \le 3 \) associated with the addition of polynomials and the multiplication of polynomials by a scalar form a vector space. Solution to Example 4 The addition of two polynomials of degree less than or equal to 3 is a polynomial of degree lass than or equal to 3. The multiplication, of a polynomial of degree less than or equal to 3, by a real number results in a polynomial of degree less than or equal to 3 Hence the set of polynomials of degree less than or equal to 3 is closed under addition and scalar multiplication (the first two conditions above). The remaining 8 rules are automatically satisfied since the polynomials are real. Example 5 Show that the set of polynomials with a degree \(n = 4 \) associated with the addition of polynomials and the multiplication of polynomials by a real number IS NOT a vector space. Solution to Example 5 The addition of two polynomials of degree 4 may not result in a polynomial of degree 4. Example: Let \(\textbf{P}(x) = -2 x^4+3x^2- 2x + 6 \) and \(\textbf{Q}(x) = 2 x^4 - 5x^2 + 10 \) \(\textbf{P}(x) + \textbf{Q}(x) = (-2 x^4+3x^2- 2x + 6) + (2 x^4 - 5x^2 + 10) = - 5x^2 - 2 x + 16 \) The result is not a polynomial of degree 4. Hence the set is not closed under addition and therefore is NOT vector space. Example 6 Show that the set of integers associated with addition and multiplication by a real number IS NOT a vector space Solution to Example 6 The multiplication of an integer by a real number may not be an integer. Example: Let \(x = - 2 \) If you multiply \(x \) by the real number \(\sqrt 3 \) the result is NOT an integer. More References and links Share — copy and redistribute the material in any medium or format for any purpose, even commercially. Adapt — remix, transform, and build upon the material for any purpose, even commercially. The licensor cannot revoke these freedoms as long as you follow the license terms. Attribution — You must give appropriate credit , provide a link to the license, and indicate if changes were made . You may do so in any reasonable manner, but not in any way that suggests the licensor endorses you or your use. ShareAlike — If you remix, transform, or build upon the material, you must distribute your contributions under the same license as the original. No additional restrictions — You may not apply legal terms or technological measures that legally restrict others from doing anything the license permits. You do not have to comply with the license for elements of the material in the public domain or where your use is permitted by an applicable exception or limitation . No warranties are given. The license may not give you all of the permissions necessary for your intended use. For example, other rights such as publicity, privacy, or moral rights may limit how you use the material. Let \(F\) be a field; for simplicity, one can assume \(F = \mathbb{R}\) or \(F = \mathbb{C}\). A vector space over \(F\) is an abelian group \((V,+)\) such that for every \(c\in F\), there is a map \(\phi_{c}: V \to V\) satisfying the following properties: For any \(v, w \in V\) and \(c\in F\), we have \[\phi_{c} (v + w) = \phi_{c} (v) + \phi_{c} (w).\] In other words, the maps \(\phi_{c}\) are group homomorphisms of \(V\). The map \(\phi_{1}: V \to V\) is just the identity map on \(V\). That is, for any \(v\in V\), one has \(\phi_{1} (v) = v\). Composition of the maps \(\phi_{c}\) corresponds to multiplication in \(F\): if \(a, b \in F\), then \[\phi_{a} \circ \phi_{b} = \phi_{ab}.\] Addition in \(F\) commutes with addition in \(V\): if \(a, b \in F\) and \(v\in V\), then \[\phi_{a+b} (v) = \phi_{a} (v) + \phi_{b} (v).\] Usually, one simply writes \(cv \) to denote \(\phi_{c} (v)\). The map \(\phi_{c}\) should be thought of as scaling the vector \(v\) by a factor of \(c\). To better understand this definition, some examples are in order: Verify that \(\mathbb{R}^2\) is a vector space over \(\mathbb{R}\) under the standard notions of vector addition and scalar multiplication. First, we must show \(\mathbb{R}^2\) is an abelian group under addition of vectors. Certainly the addition operation is associative and commutative, since addition in \(\mathbb{R}\) is associative and commutative. There is an additive identity, the zero vector \((0,0)\). For any \((a,b) \in \mathbb{R}^2\), an additive inverse exists, namely \((-a,-b)\). Thus, \((\mathbb{R}^2, +)\) is an abelian group. Next, we check that scalar multiplication \(c(a,b) = (ca, cb)\) satisfies the given properties. We have \(1(a,b) = (a,b)\), as desired. Furthermore, \[\begin{align} c\big((a,b) + (x,y)\big) &= c(a+x,b+y) \\ &= (ca+cx,cb+cy) \\ &= (ca, cb) + (cx, cy) \\ &= c(a,b) + c(x,y)\\\\ mn(a,b) &= (mna, mnb) \\ &= m(na,nb) \\ &= m\big(n(a,b)\big)\\\\ (c+d)(a,b) &= (ca+da, cb+db) \\ &= (ca, cb) + (da,db) \\ &= c(a,b) + d(a,b). \end{align}\] We conclude \(\mathbb{R}^2\), with the given addition and scalar multiplication operations, forms a vector space. \(_\square\) Other examples of vector spaces include the following: For any field \(F\), the set of \(n\)-tuples of elements in \(F\), denoted \(F^n\), is a vector space over \(F\). Vector addition and scalar multiplication are defined precisely as in the case of \(\mathbb{R}^n\) above. Taking \(F=\mathbb{Z}_2\), linear binary codes arise as vector subspaces of \(\mathbb{Z}_2^{n}\). Let \(V = C[0,1]\), the set of continuous functions \([0,1] \to \mathbb{R}\). This is a vector space over \(\mathbb{R}\). Vector addition is defined by \[(f+g)(x) = f(x) + g(x)\] and scalar multiplication by \[(cf)(x) = cf(x).\] Similarly, \(C^{k}(\mathbb{R}) \), the space of \(k\)-times continuously differentiable functions \(\mathbb{R} \to \mathbb{R}\) is a vector space over \(\mathbb{R}\). The set \(P_{n} = \big\{\sum_{i=0}^{n} a_i x^{i} \big| a_{i}\in \mathbb{R} \ \forall i \big\}\) of polynomials with real coefficients of degree \(\leq n\), with the usual polynomial addition and multiplication by a real number, is a vector space. \(\mathbb{C}\) is a vector space over \(\mathbb{R}.\) \(\mathbb{R}\) is a vector space over \(\mathbb{Q}.\) \(M_n(\mathbb{R})\), the space of matrices over \(\mathbb{R}\) of order \(n\) with the matrix addition and scalar multiplication by \(c\in \mathbb{R}\) defined by multiplying each entry of a matrix by \(c.\) A vector space is a group of objects called vectors, added collectively and multiplied by numbers, called scalars. Scalars are usually considered to be real numbers. But there are few cases of scalar multiplication by rational numbers, complex numbers, etc. with vector spaces.In this article, we have covered Vector Space Definition, Axions, Properties and others in detail.What is Vector Space?A space in mathematics comprised of vectors, that follow the associative and commutative law of addition of vectors and the associative and distributive process of multiplication of vectors by scalars is called vector space. In vector space, it consists of a set of V (elements of V are called vectors), a field F (elements of F are scalars) and the two arithmetic operationsVector Addition: It is an operation that takes two vectors u, v ∈ V, and it produces the third vector u + v ∈ VScalar Multiplication: It is an operation that takes a scalar c ∈ F and a vector v ∈ V and produces a new vector uv ∈ V.Vector Space DefinitionA space comprised of vectors, collectively with the associative and commutative law of addition of vectors and also the associative and distributive process of multiplication of vectors by scalars is called vector space. Vector Addition is an operation that takes two vectors u, v ∈ V, and it produces the third vector u + v ∈ VScalar Multiplication is an operation that takes a scalar c ∈ F and a vector v ∈ V and it produces a new vector uv ∈ VVector Space AxiomsTen axioms can define vector space. Let x, y, & z be the elements of the vector space V and a & b be the elements of the field F.1. Closed Under Addition For every element x and y in V, x + y is also in V.2. Closed Under Scalar MultiplicationFor every element x in V and scalar a in F, ax is in V.3. Commutativity of Addition For every element x and y in V, x + y = y + x.4. Associativity of AdditionFor every element x, y, and z in V, (x + y) + z = x + (y + z).5. Existence of the Additive IdentityThere exists an element in V which is denoted as 0 such that x + 0 = x, for all x in V.6. Existence of the Additive InverseFor every element x in V, there exists another element in V that we can call -x such that x + (-x) = 0.7. Existence of the Multiplicative IdentityThere exists an element in F notated as 1 so that for all x in V, 1x = x.8. Associativity of Scalar MultiplicationFor every element x in V, and for each pair of elements a and b in F, (ab)x = a(bx).9. Distribution of Elements to ScalarsFor every element a in F and every pair of elements x and y in V, a(x + y) = ax + ay.10. Distribution of Scalars to ElementsFor every element x in V, and every pair of elements a and b in F, (a + b)x = ax + bxVector Space ExamplesVarious examples of vector spaces are:Real Numbers (ℝ): Set of all real numbers forms a vector space under standard addition and scalar multiplication. For example, any two real numbers can be added together (resulting in another real number), and any real number can be multiplied by a scalar (another real number) to give another real number.Euclidean Space (ℝn): This is the classic n-dimensional vector space where vectors are represented as n-tuples of real numbers. For example, in ℝ3 (3-dimensional Euclidean space), vectors could be represented as (x, y, z), where x, y, and z are real numbers.Polynomials: Set of all polynomials with coefficients from a field (like ℝ or ℂ) forms a vector space. For example, the set of all quadratic polynomials ax2 + bx + c, where a, b, and c are real numbers, is a vector space under polynomial addition and scalar multiplication.Matrices: Set of all matrices of a fixed size (e.g., m x n matrices) with entries from a field forms a vector space. Matrices can be added together element-wise, and scalar multiplication involves multiplying each element of the matrix by a scalar.What is Difference between Vector and Vector Space?A vector is a mathematical object that has both magnitude and direction, while a vector space is a mathematical structure consisting of a set of vectors along with operations of addition and scalar multiplication, satisfying specific properties. Vectors are elements of vector spaces, providing the algebraic framework for studying linear relationships and operations.Is Zero a Vector Space?A set containing only the zero vector is called a vector space, it is also called a Zero vector Space(Trivial Vector Space). This vector satisfies all the axion of vector space and hence is called vector space.Dimension of a Vector SpaceNumber of vectors in a basis for V is called the dimension of V.For example, the dimension of Rn is n. The dimension of the vector space of polynomials in x with real coefficients having degree at most two is 3.Basis of Vector SpaceLet V be a subspace of Rn for some n. A collection B = {v1, v2, ..., vr} of vectors from V is said to be a basis for V if B is linearly independent and spans V. If either one of these criterion is not satisfied, then the collection is not a basis for V.If a collection of vectors spans V, then it contains enough vectors so that every vector in V can be written as a linear combination of those in the collection. If the collection is linearly independent, then it doesn't contain so many vectors that some become dependent on the others.Vector Addition and Scalar MultiplicationVector addition and scaler multiplication are two main concept in vector space that are explained below:Vector AdditionWhen you add two vectors, you add their corresponding components. For example, if you have two vectors v = ⟨v1, v2, v3⟩and w = ⟨w1, w2, w3⟩ their sum v+ wv+ w is ⟨v1+w1, v2+w2, v3+w3⟩. Geometrically, vector addition represents the process of moving one vector's endpoint to the other vector's endpoint, forming a new vector from the initial point of the first vector to the final point of the second vector.Scalar MultiplicationScalar multiplication involves multiplying a vector by a scalar. For example, if you have a vector ⟨v = ⟨v1, v2, v3⟩ and a scalar k, then the scalar multiple kvis ⟨kv1, kv2, kv3⟩. Geometrically, scalar multiplication stretches or compresses the vector without changing its direction, depending on whether the scalar is greater than 1 or between 0 and 1 Linear Combinations and SpanLet v1, v2,..., vr be vectors in Rn . A linear combination of these vectors is any expression of the formk1v1 + k2v2 + ... + krVrwhere the coefficients k1, k2,..., kr are scalars.Vector Space PropertiesSome important properties of vector space are:Closure under Addition: Sum of any two vectors in the vector space is also a vector in the vector space.Closure under Scalar Multiplication: Multiplying any vector in the vector space by a scalar yields another vector in the vector space.Associativity of Addition: Vector addition is associative, meaning (u + v)+ w = u + (v + w) for all vectors u, v, and w in the vector space.Commutativity of Addition: Vector addition is commutative, meaning u + v = v + u for all vectors u and v in the vector space.Existence of Additive Identity: There exists a vector, denoted by 0 or 0, called the zero vector, such that u + 0 = u for all vectors u in the vector space.Existence of Additive Inverse: For every vector u in the vector space, there exists a vector -u such that u + (-u) = 0.Distributive Properties: Scalar multiplication distributes over vector addition, meaning α(u + v) = αu + αv and (α+β)u = αu + βu for all scalars α and β, and vectors u and v in the vector space.Multiplicative Identity: Scalar 1 acts as the multiplicative identity, meaning 1·u = u for all vectors u in the vector space.A subset W of a vector space V is called a subspace of V if W is itself a vector space under the addition and scalar multiplication defined on V.Subspaces are subsets of a vector space that themselves form vector spaces. Operations of vector addition and scalar multiplication from the larger vector space are applicable to vector space. Subspaces satisfies all axion/properties of vector space. Contain the zero vectorIs closed under additionIs closed under scalar multiplicationThey can be lower-dimensional spaces within the larger vector space and can provide insights into the structure and properties of the vector space as a whole.Difference Between Vector Space and Euclidean SpaceVector Space Euclidean SpaceVector space is an abstract algebraic structure defined by vector addition and scalar multiplicationEuclidean space is a geometric space characterized by distances and coordinates.Vector space focuses on the algebraic properties of vectors and their operationsEuclidean space focuses on the geometric properties of points, lines, distances, and angles within a specific coordinate system.Vector spaces are used in linear algebra and various mathematical theories.Euclidean spaces are used extensively in geometry, physics, engineering, and other fields where spatial relationships and measurements are importantApplications of Vector SpacesWhen an object is made up of multiple components it is often useful to represent the object as a vector, with one entry per component. The examples discussed in this section involve molecules, which are made up of atoms, and text documents, which are made up of words.In some cases equations involving the objects give rise to vector equations. In other examples there are reasons to perform operations on the vectors using matrix algebra. Vector Spaces is also used in Machine Learning and its various other uses are:Data Representation: In many machine learning algorithms, data is represented as vectors. For example, images can be represented as vectors of pixel values, text documents can be represented as vectors of word counts or embeddings, and numerical data can be directly represented as vectors.Feature Vectors: Feature engineering involves creating meaningful representations of data. These representations are often in the form of feature vectors, where each feature corresponds to a dimension in the vector space. Feature vectors are used as input to machine learning models.Vector Operations: Vector operations such as addition, subtraction, dot products, and vector norms are commonly used in machine learning algorithms. For example, in clustering algorithms like k-means, vector addition and subtraction are used to calculate centroids.Linear Algebra in Models: Many machine learning models are based on linear algebra operations. For example, linear regression involves finding a line that best fits a set of data points, which can be formulated as a linear algebra problem involving vectors and matrices. A vector space is a set equipped with two operations, vector addition and scalar multiplication, satisfying certain properties. Generalizing the setup for , we have A vector space is a non-empty set equipped with two operations - vector addition “” and scalar multiplication “”- which satisfy the two closure axioms C1, C2 as well as the eight vector space axioms A1 - A8: C1 (Closure under vector addition) Given , . C2 (Closure under scalar multiplication) Given and a scalar , . For , , arbitrary vectors in , and arbitrary scalars in , A1 (Commutativity of addition) . A2 (Associativity of addition) . A3 (Existence of a zero vector) There is a vector with . A4 (Existence of additive inverses) For each , there is a vector with . A5 (Distributivity of scalar multiplication over vector addition) . A6 (Distributivity of scalar addition over scalar multiplication) . A7 (Associativity of scalar multiplication) . A8 (Scalar multiplication with 1 is the identity) . A vector space can properly be represented as a triple , to emphasize the fact that the algebraic structure depends not just on the underlying set of vectors, but on the choice of operations representing addition and scalar multiplication. Let , the space of matrices, with addition given by matrix addition and scalar multiplication as previously defined for matrices. Then is a vector space. Again, as with , the closure axioms are seen to be satisfied as a direct consequence of the definitions, while the other properties follow from Theorem thm:matalg together with direct construction of the “zero vector” , as well as additive inverses as indicated in [A4]. Before proceeding to other examples, we need to discuss an important point regarding how theorems about vector spaces are typically proven. In any system of mathematics, one operates with a certain set of assumptions, called axioms, together with various results previously proven (possibly in other areas of mathematics) and which one is allowed to assume true without further verification. In the case of vector spaces over (i.e. where the scalars are real numbers), the standing assumption is that the above list of ten properties hold for the real numbers. The fastidious reader will note that this was already assumed in the proof of Theorem thm:matalg; in fact the proof of that theorem would have been impossible without such an assumption. To illustrate how this foundational assumption applies in a different context, we consider the space Recall that i) a function is completely determined by the values it takes on the elements of its domain, and therefore ii) two functions are equal iff they have the same domain and for all elements in their common domain. So in order to show two functions and on the closed interval are equal, it suffices to verify that for all . Next recall that the sum of two functions is defined by the equality while the scalar multiple of the function is given by Equipped with addition and scalar multiplication as just defined, is a vector space. Proof We begin by verifying the two closure axioms. If , they are real-valued functions with common domain ; hence their sum is defined by the above equation, and has the same domain, making a function in . Similarly, if and , then multiplying by leaves the domain unchanged, so . The eight vector space axioms [A1] - [A8] are of two types. The third and fourth are existential - they assert the existence of the zero element and additive inverses, respectively. To verify these, one simply has to produce the candidate satisfying the requisite properties. The remaining six are universal. They involve statements which hold for all collections of vectors for which the given equality makes sense. We will verify each of the eight axioms in detail. This example, then, can be used as a template for how to proceed in other cases with verification that a proposed candidate vector space is in fact one. [A1]: For all and , [A2]: For all and , [A3]: Define by . Then for all and , [A4]: For each , define (note the different placement of parentheses on the two sides of the equation). Then for all and , [A5]: For all and , [A6]: For all and , [A7]: For all and , [A8]: For all and , Show that is a vector space, where “+” denotes matrix addition, and “” denotes scalar multiplication for matrices (in other words, verify the claim in Example example:Rmn. Hint: use the results of Theorem thm:matalg). Let denote the set of functions on the open interval (where), with addition and scalar multiplication defined as above in (eqn:function-add) and (eqn:function-scalar-mult). Show that is a vector space (Hint: Does the proof given above for work just as well for ?). The concept of a vector space is a foundational concept in mathematics, physics, and the data sciences. In this post, we first present and explain the definition of a vector space and then go on to describe properties of vector spaces. Lastly, we present a few examples of vector spaces that go beyond the usual Euclidean vectors that are often taught in introductory math and science courses.IntroductionThe concept of a vector space is a foundational concept in mathematics, physics, and the data sciences. In most introductory courses, only vectors in a Euclidean space are discussed. That is, vectors are presented as arrays of numbers:\[\boldsymbol{x} = \begin{bmatrix}1 \\ 2\end{bmatrix}\]If the array of numbers is of length two or three, than one can visualize the vector as an arrow:While this definition is adequate for most applications of vector spaces, there exists a more abstract, and therefore more sophisticated definition of vector spaces that is required to have a deeper understanding of topics in math, statistics, and machine learning. In this post, we will dig into the abstract definition for vector spaces and discuss a few of their properties. Moreover, we will look at a few examples of vector spaces outside of the usual Euclidean vectors and see how the formal definition generalizes to other mathematical constructs such as matrices and functions.Formal definitionAs we mentioned before, vectors are usually introduced as arrays of numbers, and consequently, as arrows. These arrows can be added together and scaled as depicted below:A vector space generalizes this notion of adding and scaling things that behave like Euclidean vectors.At a more rigorous mathematical level, a vector space consists of both a set of vectors $\mathcal{V}$ and a field of scalars $\mathcal{F}$ for which one can add together vectors in $\mathcal{V}$ as well as scale these vectors by elements in the field $\mathcal{F}$ according to a specific list of rules (in most cases, the field of scalars are the real numbers, $\mathbb{R}$). These rules are spelled out in the definition for a vector space:Definition 1 (vector space): Given a set of objects $\mathcal{V}$ called vectors and a field $\mathcal{F} := (C, +, \cdot, -, ^{-1}, 0, 1)$ where C is the set of elements in the field, called scalars, the tuple $(\mathcal{V}, \mathcal{F})$ is a vector space if for all $\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w} \in \mathcal{V}$ and $c, d \in C$, the following ten axioms hold:$\boldsymbol{u} + \boldsymbol{v} \in \mathcal{V}$$\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$$(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$There exists a zero vector $\boldsymbol{0} \in \mathcal{V}$ such that $\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{u}$For each $\boldsymbol{u \in \mathcal{V}}$ there exists a $\boldsymbol{u'} \in \mathcal{V}$ such that $\boldsymbol{u} + \boldsymbol{u'} = \boldsymbol{0}$. We call $\boldsymbol{u'}$ the negative of $\boldsymbol{u}$ and denote it as $-\boldsymbol{u}$The scalar multiple of $\boldsymbol{u}$ by c, denoted by $c\boldsymbol{u}$ is in $\mathcal{V}$$c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$$(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$$c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$$1\boldsymbol{u} = \boldsymbol{u}$Axioms 1-5 of the definition describe how vectors can be added together. Axioms 6-10 describe how these vectors can be scaled using the field of scalars.PropertiesThe ten axioms outlined in the definition for a vector space may seem somewhat arbitrary (at least, they did for me); however, as we will show, these axioms are sufficient for ensuring that vector spaces have all of the properties that we intuitively associate with Euclidean vectors. Specifically, from these axioms, we can derive the following properties:The zero vector is unique (Theorem 1 in the Appendix). There is only one distinct zero vector in a vector space. Notice in a Euclidean vector space, there is only one point at the origin, which represents the zero vector in Euclidean spaces.Any vector multiplied by the zero scalar is the zero vector (Theorem 2 in the Appendix).The zero scalar converts any vector into the zero vector. That is, given a vector $\boldsymbol{v}$, it holds that $0\boldsymbol{v} = \boldsymbol{0}$. This generalizes the notion of how multiplying a vector in a Euclidean space by zero should shrink the vector to the origin.The negative of a vector is unique (Theorem 3 in the Appendix). Given a vector $\boldsymbol{v}$, we denote its negative vector as $-\boldsymbol{v}$. This is analogous to each real number $x \in \mathbb{R}$ having a matching negative number $-x$ that lies $|x|$ distance from 0 on the opposite side of 0.Multiplying a negative vector by the scalar -1 produces its negative vector (Theorem 4 in the Appendix). That is, given a vector $\boldsymbol{v}$, it holds that $-1\boldsymbol{v} = -\boldsymbol{v}$. This is analogous to the fact that if you multiply any number x by -1 you get the number $-x$ that lies $|x|$ distance from 0 on the opposite side of 0.The zero vector multiplied by any scalar is the zero vector (Theorem 5 in the Appendix). The zero vector remains the zero vector despite being multiplied by any scalar. That is, $c\boldsymbol{0} = \boldsymbol{0}$ for any $c \in \mathcal{F}$. This is analogous to the fact that zero multiplied by any number remains zero.The only vector whose negative is not distinct from itself is the zero vector (Theorem 6 in the Appendix). For every vector other than the zero vector, its negative vector is a distinct vector in the vector space. For the zero vector, its negative is itself. This is analogous to the fact that for any number $x eq 0$, the number $-x$ is a distinct number from x that lies on the opposite side of 0. However, for $x = 0$, $-x = x$.Examples of vector spacesThe real numbersIt turns out that the real numbers are themselves a vector space (when equipped with standard addition and multiplication). In this vector space, the real numbers are both the vectors and the scalars! Here, the number zero acts as the zero vector. This example may be a bit trivial and silly; however, I like it because it highlights the generality of the definition of a vector space.MatricesAlthough generally not thought of as vectors, the space of real-valued matrices of a fixed size \(\mathbb{R}^{m \times n}\) form a vector space in which the matrices are vectors. Intuitively, you can add matrices together:\[\begin{bmatrix}1 & 2 \\ 3 & 4\end{bmatrix} + \begin{bmatrix}3 & 2 \\ 2 & 5\end{bmatrix} = \begin{bmatrix}4 & 4 \\ 5 & 9\end{bmatrix}\]You can also scale them:\[2\begin{bmatrix}1 & 2 \\ 3 & 4\end{bmatrix} = \begin{bmatrix}2 & 4 \\ 6 & 8\end{bmatrix}\]The zero matrix acts as the zero vector:\[\begin{bmatrix}0 & 0 \\ 0 & 0\end{bmatrix}\]This may seem a bit confusing because as we discuss in another blog post, matrices act as functions between Euclidean vector spaces. Nonetheless, matrices can form vector spaces all on their own, distinct from the vector spaces that they act upon!FunctionsSets of functions can also form vector spaces! In fact, the real power in the definition for a vector space reveals itself when dealing with functions, and the fact that some sets of functions form vector spaces lies at the foundation for many fundamental ideas in mathematics, physics, and the data sciences such as Fourier transforms and reproducing kernel Hilbert spaces.For example, the set of all continuous, real-valued functions forms a vector space. Intuitively we see that such functions act like vectors in that we can add them together:We can also scale functions. In the following figure, the function g is scaled by c:Lastly, the zero function acts as the zero vector. Here we depict the zero function, which outputs 0 for all inputs:Appendix: Proofs of properties of vector spacesTheorem 1 (Uniqueness of zero vector): Given vector space $(\mathcal{V}, \mathcal{F})$, the zero vector is unique.Proof:Assume for the sake of contradiction that there exists a vector $\boldsymbol{a}$ such that $\boldsymbol{a} eq \boldsymbol{0}$ and that $\forall \boldsymbol{v} \in \mathcal{V}$\[\boldsymbol{a} + \boldsymbol{v} = \boldsymbol{v}\]Then, this implies that\[\boldsymbol{a} + \boldsymbol{0} = \boldsymbol{0}\]However, Axiom 4 of the definition for a vector space states that if a vector is the zero-vector, it must be that $\boldsymbol{a} + \boldsymbol{0} = \boldsymbol{a}$.Since $\boldsymbol{a} eq \boldsymbol{0}$, we reach a contradiction. Therefore, there does not exist a vector $\boldsymbol{a} eq \boldsymbol{0}$ for which $\forall \boldsymbol{v} \in \mathcal{V} \ \ \boldsymbol{a} + \boldsymbol{v} = \boldsymbol{v}$. Thus, the zero-vector is unique.$\square$Theorem 2 (The product of the zero scalar and any vector is the zero vector): Given a vector space $(\mathcal{V}, \mathcal{F})$, it holds that $\forall \boldsymbol{v} \in \mathcal{V}, 0\boldsymbol{v} = \boldsymbol{0}$.Proof:Assume for the sake of contradiction that there exists a vector $\boldsymbol{a} eq \boldsymbol{0}$ such that\[0\boldsymbol{v} = \boldsymbol{a}\]Now, for any scalar $c eq 0$, we have\[\begin{align*}c\boldsymbol{v} &= (c + 0)\boldsymbol{v} \\ &= c\boldsymbol{v} + 0\boldsymbol{v} && \text{by Axiom 8} \\ &= c\boldsymbol{v} + \boldsymbol{a}\end{align*}\]Our assumption assumed that $\boldsymbol{a} eq \boldsymbol{0}$ must be false because by Theorem 1 the only vector $\boldsymbol{a}$ for which $c\boldsymbol{v} + \boldsymbol{a} = c\boldsymbol{v}$ would be true is the zero-vector.$\square$Theorem 3 (Each vector has a unique negative vector): Given a vector space $(\mathcal{V}, \mathcal{F})$ and vector $\boldsymbol{v} \in \mathcal{V}$, its negative, $-\boldsymbol{v}$, is unique. That is, $\boldsymbol{v} + \boldsymbol{a} = \boldsymbol{0} \iff \boldsymbol{a} = -\boldsymbol{v}$.Proof:We need only prove $\boldsymbol{v} + \boldsymbol{a} = \boldsymbol{0} \implies \boldsymbol{a} = -\boldsymbol{v}$. The other direction is stated in the axioms for the definition of a vector space.\[\begin{align*}\boldsymbol{v} + \boldsymbol{a} &= \boldsymbol{0} \\ \implies -\boldsymbol{v} + \boldsymbol{v} + \boldsymbol{a} &= -\boldsymbol{v} + \boldsymbol{0} \\ \implies [-\boldsymbol{v} + \boldsymbol{v}] + \boldsymbol{a} &= -\boldsymbol{v} \\ \implies \boldsymbol{0} + \boldsymbol{a} &= -\boldsymbol{v} &&\text{by Axiom 5} \\ \implies \boldsymbol{a} &= -\boldsymbol{v} && \text{by Axiom 4}\end{align*}\]$\square$Theorem 4 (Derivation of a vector’s negative): Given a vector $\boldsymbol{v} \in \mathcal{V}$, it’s negative is $(-1)\boldsymbol{v}$. That is, $-\boldsymbol{v} = (-1)\boldsymbol{v}$.Proof:\[\begin{align*}\boldsymbol{v} + (-1)\boldsymbol{v} &= (1)\boldsymbol{v} + (-1)\boldsymbol{v} && \text{by Axiom 10} \\ &= (1-1)\boldsymbol{v} && \text{by Axiom 8} \\ &= 0\boldsymbol{v} \\ &= \boldsymbol{0} && \text{by Theorem 2}\end{align*}\]Then, by Axiom 5, it must be that $(-1)\boldsymbol{v} = -\boldsymbol{v}$.$\square$Theorem 5 (The zero vector multiplied by any scalar is the zero vector): Given a vector space $(\mathcal{V}, \mathcal{F})$, it holds that $c\boldsymbol{0} = \boldsymbol{0} \iff \boldsymbol{a} = \boldsymbol{0}$.Proof: \(\begin{align*}\boldsymbol{0} + \boldsymbol{0} &= \boldsymbol{0} && \text{by Axiom 4} \\ c(\boldsymbol{0} + \boldsymbol{0}) &= c(\boldsymbol{0}) \\ c(\boldsymbol{0} + c(\boldsymbol{0}) &= c(\boldsymbol{0}) && \text{by Axiom 8}\end{align*}\)By Theorem 1, the only vector $\boldsymbol{a}$ in $\mathcal{V}$ for which $\boldsymbol{a} + \boldsymbol{v} = \boldsymbol{v}$ for all vectors $\boldsymbol{v} \in \mathcal{V}$ is the zero vector $\boldsymbol{0}$. Thus, $c\boldsymbol{0} = \boldsymbol{0}$.$\square$Theorem 6 (The zero vector is its own negative): Given a vector space $(\mathcal{V}, \mathcal{F})$, it holds that $-\boldsymbol{0} = \boldsymbol{0}$Proof:\[\begin{align*}\boldsymbol{a} + -\boldsymbol{a} &= \boldsymbol{0} && \text{by Axiom 5} \\ \boldsymbol{a} + \boldsymbol{a} &= \boldsymbol{0} && \text{assume $\boldsymbol{a} = -\boldsymbol{a}$} \\ \implies 2\boldsymbol{a} &= \boldsymbol{0} \\ \implies \boldsymbol{a} &= \boldsymbol{0} && \text{by Theorem 5}\end{align*}\]Thus, if we assume $\boldsymbol{a} = -\boldsymbol{a}$, then $\boldsymbol{a}$ must be the zero vector.$\square$ Welcome to the Mathematics Library. This Living Library is a principal hub of the LibreTexts project, which is a multi-institutional collaborative venture to develop the next generation of open-access texts to improve postsecondary education at all levels of higher learning. The LibreTexts approach is highly collaborative where an Open Access textbook environment is under constant revision by students, faculty, and outside experts to supplant conventional paper-based books. Campus BookshelvesBookshelvesLearning Objects Home is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts. Algebraic structure in linear algebra Not to be confused with Vector field. "Linear space" redirects here. For a structure in incidence geometry, see Linear space (geometry). Vector addition and scalar multiplication: a vector v (blue) is added to another vector w (red, upper illustration). Below, w is stretched by a factor of 2, yielding the sum v + 2w. In mathematics and physics, a vector space (also called a linear space) is a set whose elements, often called vectors, can be added together and multiplied ("scaled") by numbers called scalars. The operations of vector addition and scalar multiplication must satisfy certain requirements, called vector axioms. Real vector spaces and complex vector spaces are kinds of vector spaces based on different kinds of scalars: real numbers and complex numbers. Scalars can also be, more generally, elements of any field. Vector spaces generalize Euclidean vectors, which allow modeling of physical quantities (such as forces and velocity) that have not only a magnitude, but also a direction. The concept of vector spaces is fundamental for linear algebra, together with the concept of matrices, which allows computing in vector spaces. This provides a concise and synthetic way for manipulating and studying systems of linear equations. Vector spaces are characterized by their dimension, which, roughly speaking, specifies the number of independent directions in the space. This means that, for two vector spaces over a given field and with the same dimension, the properties that depend only on the vector-space structure are exactly the same (technically the vector spaces are isomorphic). A vector space is finite-dimensional if its dimension is a natural number. Otherwise, it is infinite-dimensional, and its dimension is an infinite cardinal. Finite-dimensional vector spaces occur naturally in geometry and related areas. Infinite-dimensional vector spaces occur in many areas of mathematics. For example, polynomial rings are countably infinite-dimensional vector spaces, and many function spaces have the cardinality of the continuum as a dimension. Many vector spaces that are considered in mathematics are also endowed with other structures. This is the case of algebras, which include field extensions, polynomial rings, associative algebras and Lie algebras. This is also the case of topological vector spaces, which include function spaces, inner product spaces, normed spaces, Hilbert spaces and Banach spaces. Algebraic structures Group-like Group Semigroup / Monoid Rack and quandle Quasigroup and loop Abelian group Magma Lie group Group theory Ring-like Ring Rng Semiring Near-ring Commutative ring Domain Integral domain Field Division ring Lie ring Ring theory Lattice-like Lattice Semilattice Complemented lattice Total order Heyting algebra Boolean algebra Map of lattices Lattice theory Module-like Module Group with operators Vector space Linear algebra Algebra-like Algebra Associative Non-associative Composition algebra Lie algebra Graded Bialgebra Hopf algebra vte In this article, vectors are represented in boldface to distinguish them from scalars.[nb 1][1] A vector space over a field F is a non-empty set V together with a binary operation and a binary function that satisfy the eight axioms listed below. In this context, the elements of V are commonly called vectors, and the elements of F are called scalars.[2] The binary operation, called vector addition or simply addition assigns to any two vectors v and w in V a third vector in V which is commonly written as v + w, and called the sum of these two vectors. The binary function, called scalar multiplication, assigns to any scalar a in F and any vector v in V another vector in V, which is denoted av.[nb 2] To have a vector space, the eight following axioms must be satisfied for every u, v and w in V, and a and b in F.[3] Axiom Statement Associativity of vector addition u + (v + w) = (u + v) + w Commutativity of vector addition u + v = v + u Identity element of vector addition There exists an element 0 ∈ V, called the zero vector, such that v + 0 = v for all v ∈ V. Inverse elements of vector addition For every v ∈ V, there exists an element −v ∈ V, called the additive inverse of v, such that v + (−v) = 0. Compatibility of scalar multiplication with field multiplication a(bv) = (ab)v [nb 3] Identity element of scalar multiplication 1v = v, where 1 denotes the multiplicative identity in F. Distributivity of scalar multiplication with respect to vector addition a(u + v) = au + av Distributivity of scalar multiplication with respect to field addition (a + b)v = av + bv When the scalar field is the real numbers, the vector space is called a real vector space, and when the scalar field is the complex numbers, the vector space is called a complex vector space.[4] These two cases are the most common ones, but vector spaces with scalars in an arbitrary field F are also commonly considered. Such a vector space is called an F-vector space or a vector space over F.[5] An equivalent definition of a vector space can be given, which is much more concise but less elementary: the first four axioms (related to vector addition) say that a vector space is an abelian group under addition, and the four remaining axioms (related to the scalar multiplication) say that this operation defines a ring homomorphism from the field F into the endomorphism ring of this group.[6] Subtraction of two vectors can be defined as v − w = v + (− w) . {\displaystyle \mathbf {v} -\mathbf {w} =\mathbf {v} +(-\mathbf {w}).} Direct consequences of the axioms include that, for every s ∈ F {\displaystyle s\in F} and v ∈ V , {\displaystyle \mathbf {v} \in V,} one has 0 v = 0 , {\displaystyle 0\mathbf {v} =\mathbf {0} ,} s 0 = 0 , {\displaystyle s\mathbf {0} =\mathbf {0} ,} (− 1) v = − v , {\displaystyle (-1)\mathbf {v} =-\mathbf {v} ,} s v = 0 {\displaystyle s\mathbf {v} =\mathbf {0} } implies s = 0 {\displaystyle s=0} or v = 0 . {\displaystyle \mathbf {v} =\mathbf {0} .} Even more concisely, a vector space is a module over a field.[7] A vector v in R2 (blue) expressed in terms of different bases: using the standard basis of R2: v = xe1 + ye2 (black), and using a different, non-orthogonal basis: v = f1 + f2 (red). Linear combination Given a set G of elements of a F-vector space V, a linear combination of elements of G is an element of V of the form a 1 g 1 + a 2 g 2 + ⋯ + a k g k , {\displaystyle a_{1}\mathbf {g} _{1}+a_{2}\mathbf {g} _{2}+\cdots +a_{k}\mathbf {g} _{k},} where a 1 , … , a k ∈ F {\displaystyle a_{1},\ldots ,a_{k}\in F} and g 1 , … , g k ∈ G . {\displaystyle \mathbf {g} _{1},\ldots ,\mathbf {g} _{k}\in G.} The scalars a 1 , … , a k {\displaystyle a_{1},\ldots ,a_{k}} are called the coefficients of the linear combination.[8] Linear independence The elements of a subset G of a F-vector space V are said to be linearly independent if no element of G can be written as a linear combination of the other elements of G. Equivalently, they are linearly independent if two linear combinations of elements of G define the same element of V if and only if they have the same coefficients. Also equivalently, they are linearly independent if a linear combination results in the zero vector if and only if all its coefficients are zero.[9] Linear subspace A linear subspace or vector subspace W of a vector space V is a non-empty subset of V that is closed under vector addition and scalar multiplication; that is, the sum of two elements of W and the product of an element of W by a scalar belong to W.[10] This implies that every linear combination of elements of W belongs to W. A linear subspace is a vector space for the induced addition and scalar multiplication; this means that the closure property implies that the axioms of a vector space are satisfied.[11]The closure property also implies that every intersection of linear subspaces is a linear subspace.[11] Linear span Given a subset G of a vector space V, the linear span or simply the span of G is the smallest linear subspace of V that contains G, in the sense that it is the intersection of all linear subspaces that contain G. The span of G is also the set of all linear combinations of elements of G. If W is the span of G, one says that G spans or generates W, and that G is a spanning set or a generating set of W.[12] Basis and dimension A subset of a vector space is a basis if its elements are linearly independent and span the vector space.[13] Every vector space has at least one basis, or many in general (see Basis (linear algebra) § Proof that every vector space has a basis).[14] Moreover, all bases of a vector space have the same cardinality, which is called the dimension of the vector space (see Dimension theorem for vector spaces).[15] This is a fundamental property of vector spaces, which is detailed in the remainder of the section. Bases are a fundamental tool for the study of vector spaces, especially when the dimension is finite. In the infinite-dimensional case, the existence of infinite bases, often called Hamel bases, depends on the axiom of choice. It follows that, in general, no base can be explicitly described.[16] For example, the real numbers form an infinite-dimensional vector space over the rational numbers, for which no specific basis is known. Consider a basis (b 1 , b 2 , … , b n) {\displaystyle (\mathbf {b} _{1},\mathbf {b} _{2},\ldots ,\mathbf {b} _{n})} of a vector space V of dimension n over a field F. The definition of a basis implies that every v ∈ V {\displaystyle \mathbf {v} \in V} may be written v = a 1 b 1 + ⋯ + a n b n , {\displaystyle \mathbf {v} =a_{1}\mathbf {b} _{1}+\cdots +a_{n}\mathbf {b} _{n},} with a 1 , … , a n {\displaystyle a_{1},\dots ,a_{n}} in F, and that this decomposition is unique. The scalars a 1 , … , a n {\displaystyle a_{1},\ldots ,a_{n}} are called the coordinates of v on the basis. They are also said to be the coefficients of the decomposition of v on the basis. One also says that the n-tuple of the coordinates is the coordinate vector of v on the basis, since the set F n {\displaystyle F^{n}} of the n-tuples of elements of F is a vector space for componentwise addition and scalar multiplication, whose dimension is n. The one-to-one correspondence between vectors and their coordinate vectors maps vector addition to vector addition and scalar multiplication to scalar multiplication. It is thus a vector space isomorphism, which allows translating reasonings and computations on vectors into reasonings and computations on their coordinates.[17] Vector spaces stem from affine geometry, via the introduction of coordinates in the plane or three-dimensional space. Around 1636, French mathematicians René Descartes and Pierre de Fermat founded analytic geometry by identifying solutions to an equation of two variables with points on a plane curve.[18] To achieve geometric solutions without using coordinates, Bolzano introduced, in 1804, certain operations on points, lines, and planes, which are predecessors of vectors.[19] Möbius (1827) introduced the notion of barycentric coordinates.[20] Bellavitis (1833) introduced an equivalence relation on directed line segments that share the same length and direction which he called equipollence.[21] A Euclidean vector is then an equivalence class of that relation.[22] Vectors were reconsidered with the presentation of complex numbers by Argand and Hamilton and the inception of quaternions by the latter.[23] They are elements in R2 and R4; treating them using linear combinations goes back to Laguerre in 1867, who also defined systems of linear equations. In 1857, Cayley introduced the matrix notation which allows for harmonization and simplification of linear maps. Around the same time, Grassmann studied the barycentric calculus initiated by Möbius. He envisaged sets of abstract objects endowed with operations.[24] In his work, the concepts of linear independence and dimension, as well as scalar products are present. Grassmann's 1844 work exceeds the framework of vector spaces as well since his considering multiplication led him to what are today called algebras. Italian mathematician Peano was the first to give the modern definition of vector spaces and linear maps in 1888,[25] although he called them "linear systems".[26] Peano's axiomatization allowed for vector spaces with infinite dimension, but Peano did not develop that theory further. In 1897, Salvatore Pincherle adopted Peano's axioms and made initial inroads into the theory of infinite-dimensional vector spaces.[27] An important development of vector spaces is due to the construction of function spaces by Henri Lebesgue. This was later formalized by Banach and Hilbert, around 1920.[28] At that time, algebra and the new field of functional analysis began to interact, notably with key concepts such as spaces of p-integrable functions and Hilbert spaces.[29] Main article: Examples of vector spaces Vector addition: the sum v + w (black) of the vectors v (blue) and w (red) is shown.Scalar multiplication: the multiples −v and 2w are shown. The first example of a vector space consists of arrows in a fixed plane, starting at one fixed point. This is used in physics to describe forces or velocities.[30] Given any two such arrows, v and w, the parallelogram spanned by these two arrows contains one diagonal arrow that starts at the origin, too. This new arrow is called the sum of the two arrows, and is denoted v + w. In the special case of two arrows on the same line, their sum is the arrow on this line whose length is the sum or the difference of the lengths, depending on whether the arrows have the same direction. Another operation that can be done with arrows is scaling: given any positive real number a, the arrow that has the same direction as v, but is dilated or shrunk by multiplying its length by a, is called multiplication of v by a. It is denoted av. When a is negative, av is defined as the arrow pointing in the opposite direction instead.[31] The following shows a few examples: if a = 2, the resulting vector aw has the same direction as w, but is stretched to the double length of w (the second image). Equivalently, 2w is the sum w + w. Moreover, (−1)v = −v has the opposite direction and the same length as v (blue vector pointing down in the second image). A second key example of a vector space is provided by pairs of real numbers x and y. The order of the components x and y is significant, so such a pair is also called an ordered pair. Such a pair is written as (x, y). The sum of two such pairs and the multiplication of a pair with a number is defined as follows:[32] (x 1 , y 1) + (x 2 , y 2) = (x 1 + x 2 , y 1 + y 2) , a (x , y) = (a x , a y) . {\displaystyle {\begin{aligned}(x_{1},y_{1})+(x_{2},y_{2})&=(x_{1}+x_{2},y_{1}+y_{2}),\\a(x,y)&=(ax,ay).\end{aligned}}} The first example above reduces to this example if an arrow is represented by a pair of Cartesian coordinates of its endpoint. The simplest example of a vector space over a field F is the field F itself with its addition viewed as vector addition and its multiplication viewed as scalar multiplication. More generally, all n-tuples (sequences of length n) (a 1 , a 2 , … , a n) {\displaystyle (a_{1},a_{2},\dots ,a_{n})} of elements ai of F form a vector space that is usually denoted Fn and called a coordinate space.[33] The case n = 1 is the above-mentioned simplest example, in which the field F is also regarded as a vector space over itself. The case F = R and n = 2 (so R2) reduces to the previous example. The set of complex numbers C, numbers that can be written in the form x + iy for real numbers x and y where i is the imaginary unit, form a vector space over the reals with the usual addition and multiplication: (x + iy) + (a + ib) = (x + a) + i(y + b) and c ⋅ (x + iy) = (c ⋅ x) + i(c ⋅ y) for real numbers x, y, a, b and c. The various axioms of a vector space follow from the fact that the same rules hold for complex number arithmetic. The example of complex numbers is essentially the same as (that is, it is isomorphic to) the vector space of ordered pairs of real numbers mentioned above: if we think of the complex number x + i y as representing the ordered pair (x, y) in the complex plane then we see that the rules for addition and scalar multiplication correspond exactly to those in the earlier example. More generally, field extensions provide another class of examples of vector spaces, particularly in algebra and algebraic number theory: a field F containing a smaller field E is an E-vector space, by the given multiplication and addition operations of F.[34] For example, the complex numbers are a vector space over R, and the field extension Q (i 5) {\displaystyle \mathbf {Q} (i{\sqrt {5}})} is a vector space over Q. Main article: Function space Addition of functions: the sum of the sine and the exponential function is sin + exp : R → R {\displaystyle \sin +\exp :\mathbb {R} \to \mathbb {R} } with (sin + exp) (x) = sin ⁡ (x) + exp ⁡ (x) {\displaystyle (\sin +\exp)(x)=\sin(x)+\exp(x)} . Functions from any fixed set Ω to a field F also form vector spaces, by performing addition and scalar multiplication pointwise. That is, the sum of two functions f and g is the function (f + g) {\displaystyle (f+g)} given by (f + g) (w) = f (w) + g (w) , {\displaystyle (f+g)(w)=f(w)+g(w),} and similarly for multiplication. Such function spaces occur in many geometric situations, when Ω is the real line or an interval, or other subsets of R. Many notions in topology and analysis, such as continuity, integrability or differentiability are well-behaved with respect to linearity: sums and scalar multiples of functions possessing such a property still have that property.[35] Therefore, the set of such functions are vector spaces, whose study belongs to functional analysis. Main articles: Linear equation, Linear differential equation, and Systems of linear equations Systems of homogeneous linear equations are closely tied to vector spaces.[36] For example, the solutions of a + 3 b + c = 0 4 a + 2 b + 2 c = 0 {\displaystyle {\begin{alignedat}{9}&&a\,&&+\,3b\,&\,+&\,&c&\,=0\\4&&a\,&&+\,2b\,&\,+&\,2&c&\,=0\\\end{alignedat}}} are given by triples with arbitrary a , {\displaystyle a,} b = a / 2 , {\displaystyle b=a/2,} and c = − 5 a / 2 . {\displaystyle c=-5a/2.} They form a vector space: sums and scalar multiples of such triples still satisfy the same ratios of the three variables; thus they are solutions, too. Matrices can be used to condense multiple linear equations as above into one vector equation, namely A x = 0 , {\displaystyle A\mathbf {x} =\mathbf {0} ,} where A = [1 3 1 4 2 2] {\displaystyle A={\begin{bmatrix}1&3&1\\4&2&2\end{bmatrix}}} is the matrix containing the coefficients of the given equations, x {\displaystyle \mathbf {x} } is the vector (a , b , c) , {\displaystyle (a,b,c),} A x {\displaystyle A\mathbf {x} } denotes the matrix product, and 0 = (0 , 0) {\displaystyle \mathbf {0} =(0,0)} is the zero vector. In a similar vein, the solutions of homogeneous linear differential equations form vector spaces. For example, f ′ ′ (x) + 2 f ′ (x) + f (x) = 0 {\displaystyle f^{\prime \prime }(x)+2f^{\prime }(x)+f(x)=0} yields f (x) = a e − x + b x e − x , {\displaystyle f(x)=ae^{-x}+bxe^{-x},} where a {\displaystyle a} and b {\displaystyle b} are arbitrary constants, and e x {\displaystyle e^{x}} is the natural exponential function. Main article: Linear map The relation of two vector spaces can be expressed by linear map or linear transformation. They are functions that reflect the vector space structure, that is, they preserve sums and scalar multiplication: f (v + w) = f (v) + f (w) , f (a ⋅ v) = a ⋅ f (v) {\displaystyle {\begin{aligned}f(\mathbf {v} +\mathbf {w})&=f(\mathbf {v})+f(\mathbf {w}),\\f(a\cdot \mathbf {v})&=a\cdot f(\mathbf {v})\end{aligned}}} for all v {\displaystyle \mathbf {v} } and w {\displaystyle \mathbf {w} } in V , {\displaystyle V,} all a {\displaystyle a} in F . {\displaystyle F.} [37] An isomorphism is a linear map f : V → W such that there exists an inverse map g : W → V, which is a map such that the two possible compositions f ∘ g : W → W and g ∘ f : V → V are identity maps. Equivalently, f is both one-to-one (injective) and onto (surjective).[38] If there exists an isomorphism between V and W, the two spaces are said to be isomorphic; they are then essentially identical as vector spaces, since all identities holding in V are, via f, transported to similar ones in W, and vice versa via g. Describing an arrow vector v by its coordinates x and y yields an isomorphism of vector spaces. For example, the arrows in the plane and the ordered pairs of numbers vector spaces in the introduction above (see § Examples) are isomorphic: a planar arrow v departing at the origin of some (fixed) coordinate system can be expressed as an ordered pair by considering the x- and y-component of the arrow, as shown in the image at the right. Conversely, given a pair (x, y), the arrow going by x to the right (or to the left, if x is negative), and y up (down, if y is negative) turns back the arrow v.[39] Linear maps V → W between two vector spaces form a vector space HomF(V, W), also denoted L(V, W), or (V, W).[40] The space of linear maps from V to F is called the dual vector space, denoted V∗.[41] Via the injective natural map V → V∗∗, any vector space can be embedded into its bidual; the map is an isomorphism if and only if the space is finite-dimensional.[42] Once a basis of V is chosen, linear maps f : V → W are completely determined by specifying the images of the basis vectors, because any element of V is expressed uniquely as a linear combination of them.[43] If dim V = dim W, a 1-to-1 correspondence between fixed bases of V and W gives rise to a linear map that maps any basis element of V to the corresponding basis element of W. It is an isomorphism, by its very definition.[44] Therefore, two vector spaces over a given field are isomorphic if their dimensions agree and vice versa. Another way to express this is that any vector space over a given field is completely classified (up to isomorphism) by its dimension, a single number. In particular, any n-dimensional F-vector space V is isomorphic to Fn. However, there is no "canonical" or preferred isomorphism; an isomorphism φ : Fn → V is equivalent to the choice of a basis of V, by mapping the standard basis of Fn to V, via φ. Main articles: Matrix and Determinant A typical matrix Matrices are a useful notion to encode linear maps.[45] They are written as a rectangular array of scalars as in the image at the right. Any m-by-n matrix A {\displaystyle A} gives rise to a linear map from Fn to Fm, by the following x = (x 1 , x 2 , … , x n) ↦ (∑ j = 1 n a 1 j x j , ∑ j = 1 n a 2 j x j , … , ∑ j = 1 n a m j x j) , {\displaystyle \mathbf {x} =(x_{1},x_{2},\ldots ,x_{n})\mapsto \left(\sum _{j=1}^{n}a_{1j}x_{j},\sum _{j=1}^{n}a_{2j}x_{j},\ldots ,\sum _{j=1}^{n}a_{mj}x_{j}\right),} where ∑ {\textstyle \sum } denotes summation, or by using the matrix multiplication of the matrix A {\displaystyle A} with the coordinate vector x {\displaystyle \mathbf {x} } : x ↦ A x . {\displaystyle \mathbf {x} \mapsto A\mathbf {x} .} Moreover, after choosing bases of V and W, any linear map f : V → W is uniquely represented by a matrix via this assignment.[46] The volume of this parallelepiped is the absolute value of the determinant of the 3-by-3 matrix formed by the vectors r1, r2, and r3. The determinant det (A) of a square matrix A is a scalar that tells whether the associated map is an isomorphism or not: to be so it is sufficient and necessary that the determinant is nonzero.[47] The linear transformation of Rn corresponding to a real n-by-n matrix is orientation preserving if and only if its determinant is positive. Main article: Eigenvalues and eigenvectors Endomorphisms, linear maps f : V → V, are particularly important since in this case vectors v can be compared with their image under f, f(v). Any nonzero vector v satisfying λv = f(v), where λ is a scalar, is called an eigenvector of f with eigenvalue λ.[48] Equivalently, v is an element of the kernel of the difference f − λ · Id (where Id is the identity map V → V). If V is finite-dimensional, this can be rephrased using determinants: f having eigenvalue λ is equivalent to det (f − λ ⋅ Id) = 0. {\displaystyle \det(f-\lambda \cdot \operatorname {Id})=0.} By spelling out the definition of the determinant, the expression on the left hand side can be seen to be a polynomial function in λ, called the characteristic polynomial of f.[49] If the field F is large enough to contain a zero of this polynomial (which automatically happens for F algebraically closed, such as F = C) any linear map has at least one eigenvector. The vector space V may or may not possess an eigenbasis, a basis consisting of eigenvectors. This phenomenon is governed by the Jordan canonical form of the map.[50] The set of all eigenvectors corresponding to a particular eigenvalue of f forms a vector space known as the eigenspace corresponding to the eigenvalue (and f) in question. In addition to the above concrete examples, there are a number of standard linear algebraic constructions that yield vector spaces related to given ones. Main articles: Linear subspace and Quotient vector space A line passing through the origin (blue, thick) in R3 is a linear subspace. It is the intersection of two planes (green and yellow). A nonempty subset W {\displaystyle W} of a vector space V {\displaystyle V} that is closed under addition and scalar multiplication (and therefore contains the 0 {\displaystyle \mathbf {0} } -vector of V {\displaystyle V}) is called a linear subspace of V {\displaystyle V} , or simply a subspace of V {\displaystyle V} , when the ambient space is unambiguously a vector space.[51][nb 4] Subspaces of V {\displaystyle V} are vector spaces (over the same field) in their own right. The intersection of all subspaces containing a given set S {\displaystyle S} of vectors is called its span, and it is the smallest subspace of V {\displaystyle V} containing the set S {\displaystyle S} . Expressed in terms of elements, the span is the subspace consisting of all the linear combinations of elements of S {\displaystyle S} .[52] Linear subspace of dimension 1 and 2 are referred to as a line (also vector line), and a plane respectively. If W is an n-dimensional vector space, any subspace of dimension 1 less, i.e., of dimension n − 1 {\displaystyle n-1} is called a hyperplane.[53] The counterpart to subspaces are quotient vector spaces.[54] Given any subspace W ⊆ V {\displaystyle W\subseteq V} , the quotient space V / W {\displaystyle V/W} (" V {\displaystyle V} modulo W {\displaystyle W} ") is defined as follows: as a set, it consists of v + W = { v + w : w ∈ W } , {\displaystyle \mathbf {v} +W=\{\mathbf {v} +\mathbf {w} :\mathbf {w} \in W\},} where v {\displaystyle \mathbf {v} } is an arbitrary vector in V {\displaystyle V} . The sum of two such elements v 1 + W {\displaystyle \mathbf {v} _{1}+W} and v 2 + W {\displaystyle \mathbf {v} _{2}+W} is (v 1 + v 2) + W {\displaystyle \left(\mathbf {v} _{1}+\mathbf {v} _{2}\right)+W} , and scalar multiplication is given by a · (v + W) = (a · v) + W {\displaystyle a\cdot (\mathbf {v} +W)=(a\cdot \mathbf {v})+W} . The key point in this definition is that v 1 + W = v 2 + W {\displaystyle \mathbf {v} _{1}+W=\mathbf {v} _{2}+W} if and only if the difference of v 1 {\displaystyle \mathbf {v} _{1}} and v 2 {\displaystyle \mathbf {v} _{2}} lies in W {\displaystyle W} .[nb 5] This way, the quotient space "forgets" information that is contained in the subspace W {\displaystyle W} . The kernel ker ⁡ (f) {\displaystyle \ker(f)} of a linear map f : V → W {\displaystyle f:V\to W} consists of vectors v {\displaystyle \mathbf {v} } that are mapped to 0 {\displaystyle \mathbf {0} } in W {\displaystyle W} .[55] The kernel and the image im ⁡ (f) = { f (v) : v ∈ V } {\displaystyle \operatorname {im} (f)=\{f(\mathbf {v}):\mathbf {v} \in V\}} are subspaces of V {\displaystyle V} and W {\displaystyle W} , respectively.[56] An important example is the kernel of a linear map x ↦ A x {\displaystyle \mathbf {x} \mapsto A\mathbf {x} } for some fixed matrix A {\displaystyle A} . The kernel of this map is the subspace of vectors x {\displaystyle \mathbf {x} } such that A x = 0 {\displaystyle A\mathbf {x} =\mathbf {0} } , which is precisely the set of solutions to the system of homogeneous linear equations belonging to A {\displaystyle A} . This concept also extends to linear differential equations a 0 f + a 1 d f d x + a 2 d 2 f d x 2 + ⋯ + a n d n f d x n = 0 , {\displaystyle a_{0}f+a_{1}{\frac {df}{dx}}+a_{2}{\frac {d^{2}f}{dx^{2}}}+\cdots +a_{n}{\frac {d^{n}f}{dx^{n}}}=0,} where the coefficients a i {\displaystyle a_{i}} are functions in x , {\displaystyle x,} too. In the corresponding map f ↦ D (f) = ∑ i = 0 n a i d i f d x i , {\displaystyle f\mapsto D(f)=\sum _{i=0}^{n}a_{i}{\frac {d^{i}f}{dx^{i}}},} the derivatives of the function f {\displaystyle f} appear linearly (as opposed to f ′ ′ (x) 2 {\displaystyle f^{\prime \prime }(x)^{2}} , for example). Since differentiation is a linear procedure (that is, (f + g) ′ = f ′ + g ′ {\displaystyle (f+g)^{\prime }=f^{\prime }+g^{\prime }} and (c ⋅ f) ′ = c ⋅ f ′ {\displaystyle (c\cdot f)^{\prime }=c\cdot f^{\prime }} for a constant c {\displaystyle c}) this assignment is linear, called a linear differential operator. In particular, the solutions to the differential equation D (f) = 0 {\displaystyle D(f)=0} form a vector space (over R or C).[57] The existence of kernels and images is part of the statement that the category of vector spaces (over a fixed field F {\displaystyle F}) is an abelian category, that is, a corpus of mathematical objects and structure-preserving maps between them (a category) that behaves much like the category of abelian groups.[58] Because of this, many statements such as the first isomorphism theorem (also called rank–nullity theorem in matrix-related terms) V / ker ⁡ (f) ≡ im ⁡ (f) {\displaystyle V/\ker(f)\;\equiv \;\operatorname {im} (f)} and the second and third isomorphism theorem can be formulated and proven in a way very similar to the corresponding statements for groups. Main articles: Direct product and Direct sum of modules The direct product of vector spaces and the direct sum of vector spaces are two ways of combining an indexed family of vector spaces into a new vector space. The direct product ∏ i ∈ I V i {\displaystyle \textstyle {\prod _{i\in I}V_{i}}} of a family of vector spaces V i {\displaystyle V_{i}} consists of the set of all tuples (v i) i ∈ I {\displaystyle \left(\mathbf {v} _{i}\right)_{i\in I}} , which specify for each index i {\displaystyle i} in some index set I {\displaystyle I} an element v i {\displaystyle \mathbf {v} _{i}} of V i {\displaystyle V_{i}} .[59] Addition and scalar multiplication is performed componentwise. A variant of this construction is the direct sum ⨁ i ∈ I V i {\textstyle \bigoplus _{i\in I}V_{i}} (also called coproduct and denoted ∐ i ∈ I V i {\textstyle \coprod _{i\in I}V_{i}}), where only tuples with finitely many nonzero vectors are allowed. If the index set I {\displaystyle I} is finite, the two constructions agree, but in general they are different. Main article: Tensor product of vector spaces The tensor product V ⊗ F W , {\displaystyle V\otimes _{F}W,} or simply V ⊗ W , {\displaystyle V\otimes W,} of two vector spaces V {\displaystyle V} and W {\displaystyle W} is one of the central notions of multilinear algebra which deals with extending notions such as linear maps to several variables. A map g : V × W → X {\displaystyle g:V\times W\to X} from the Cartesian product V × W {\displaystyle V\times W} is called bilinear if g {\displaystyle g} is linear in both variables v {\displaystyle \mathbf {v} } and w . {\displaystyle \mathbf {w} .} That is to say, for fixed w {\displaystyle \mathbf {w} } the map v ↦ g (v , w) {\displaystyle \mathbf {v} \mapsto g(\mathbf {v} ,\mathbf {w})} is linear in the sense above and likewise for fixed v . {\displaystyle \mathbf {v} .} Commutative diagram depicting the universal property of the tensor product The tensor product is a particular vector space that is a universal recipient of bilinear maps g , {\displaystyle g,} as follows. It is defined as the vector space consisting of finite (formal) sums of symbols called tensors v 1 ⊗ w 1 + v 2 ⊗ w 2 + ⋯ + v n ⊗ w n , {\displaystyle \mathbf {v} _{1}\otimes \mathbf {w} _{1}+\mathbf {v} _{2}\otimes \mathbf {w} _{2}+\cdots +\mathbf {v} _{n}\otimes \mathbf {w} _{n},} subject to the rules[60] a ⋅ (v ⊗ w) = (a ⋅ v) ⊗ w = v ⊗ (a ⋅ w) , where a is a scalar (v 1 + v 2) ⊗ w = v 1 ⊗ w + v 2 ⊗ w v ⊗ (w 1 + w 2) = v ⊗ w 1 + v ⊗ w 2 . {\displaystyle {\begin{alignedat}{6}a\cdot (\mathbf {v} \otimes \mathbf {w})~&=~(a\cdot \mathbf {v})\otimes \mathbf {w} ~=~\mathbf {v} \otimes (a\cdot \mathbf {w}),&&~~{\text{ where }}a{\text{ is a scalar}}\\(\mathbf {v} _{1}+\mathbf {v} _{2})\otimes \mathbf {w} ~&=~\mathbf {v} _{1}\otimes \mathbf {w} +\mathbf {v} _{2}\otimes \mathbf {w} &&\\\mathbf {v} \otimes (\mathbf {w} _{1}+\mathbf {w} _{2})~&=~\mathbf {v} \otimes \mathbf {w} _{1}+\mathbf {v} \otimes \mathbf {w} _{2}.&&\\\end{alignedat}}} These rules ensure that the map f {\displaystyle f} from the V × W {\displaystyle V\times W} to V ⊗ W {\displaystyle V\otimes W} that maps a tuple (v , w) {\displaystyle (\mathbf {v} ,\mathbf {w})} to v ⊗ w {\displaystyle \mathbf {v} \otimes \mathbf {w} } is bilinear. The universality states that given any vector space X {\displaystyle X} and any bilinear map g : V × W → X , {\displaystyle g:V\times W\to X,} there exists a unique map u , {\displaystyle u,} shown in the diagram with a dotted arrow, whose composition with f {\displaystyle f} equals g : {\displaystyle g:} u (v ⊗ w) = g (v , w) . {\displaystyle u(\mathbf {v} \otimes \mathbf {w})=g(\mathbf {v} ,\mathbf {w}).} [61] This is called the universal property of the tensor product, an instance of the method—much used in advanced abstract algebra—to indirectly define objects by specifying maps from or to this object. From the point of view of linear algebra, vector spaces are completely understood insofar as any vector space over a given field is characterized, up to isomorphism, by its dimension. However, vector spaces per se do not offer a framework to deal with the question—crucial to analysis—whether a sequence of functions converges to another function. Likewise, linear algebra is not adapted to deal with infinite series, since the addition operation allows only finitely many terms to be added. Therefore, the needs of functional analysis require considering additional structures.[62] A vector space may be given a partial order ≤ , {\displaystyle \,\leq ,\,} under which some vectors can be compared.[63] For example, n {\displaystyle n} -dimensional real space R n {\displaystyle \mathbf {R} ^{n}} can be ordered by comparing its vectors componentwise. Ordered vector spaces, for example Riesz spaces, are fundamental to Lebesgue integration, which relies on the ability to express a function as a difference of two positive functions f = f + − f − . {\displaystyle f=f^{+}-f^{-}.} where f + {\displaystyle f^{+}} denotes the positive part of f {\displaystyle f} and f − {\displaystyle f^{-}} the negative part.[64] Main articles: Normed vector space and Inner product space "Measuring" vectors is done by specifying a norm, a datum which measures lengths of vectors, or by an inner product, which measures angles between vectors. Norms and inner products are denoted | v | {\displaystyle |\mathbf {v} |} and ⟨ v , w ⟩ , {\displaystyle \langle \mathbf {v} ,\mathbf {w} \rangle ,} respectively. The datum of an inner product entails that lengths of vectors can be defined too, by defining the associated norm | v | := ⟨ v , v ⟩ . {\textstyle |\mathbf {v} |:={\sqrt {\langle \mathbf {v} ,\mathbf {v} \rangle }}.} Vector spaces endowed with such data are known as normed vector spaces and inner product spaces, respectively.[65] Coordinate space F n {\displaystyle F^{n}} can be equipped with the standard dot product: ⟨ x , y ⟩ = x ⋅ y = x 1 y 1 + ⋯ + x n y n . {\displaystyle \langle \mathbf {x} ,\mathbf {y} \rangle =\mathbf {x} \cdot \mathbf {y} =x_{1}y_{1}+\cdots +x_{n}y_{n}.} In R 2 , {\displaystyle \mathbf {R} ^{2},} this reflects the common notion of the angle between two vectors x {\displaystyle \mathbf {x} } and y , {\displaystyle \mathbf {y} ,} by the law of cosines: x ⋅ y = cos ⁡ (∠ (x , y)) ⋅ | x | ⋅ | y | . {\displaystyle \mathbf {x} \cdot \mathbf {y} =\cos \left(\angle (\mathbf {x} ,\mathbf {y})\right)\cdot |\mathbf {x} |\cdot |\mathbf {y} |.} Because of this, two vectors satisfying ⟨ x , y ⟩ = 0 {\displaystyle \langle \mathbf {x} ,\mathbf {y} \rangle =0} are called orthogonal. An important variant of the standard dot product is used in Minkowski space: R 4 {\displaystyle \mathbf {R} ^{4}} endowed with the Lorentz product[66] ⟨ x | y ⟩ = x 1 y 1 + x 2 y 2 + x 3 y 3 − x 4 y 4 . {\displaystyle \langle \mathbf {x} |\mathbf {y} \rangle =x_{1}y_{1}+x_{2}y_{2}+x_{3}y_{3}-x_{4}y_{4}.} In contrast to the standard dot product, it is not positive definite: ⟨ x | x ⟩ {\displaystyle \langle \mathbf {x} |\mathbf {x} \rangle } also takes negative values, for example, for x = (0 , 0 , 0 , 1) . {\displaystyle \mathbf {x} =(0,0,0,1).} Singling out the fourth coordinate—corresponding to time, as opposed to three space-dimensions—makes it useful for the mathematical treatment of special relativity. Note that in other conventions time is often written as the first, or "zeroeth" component so that the Lorentz product is written ⟨ x | y ⟩ = − x 0 y 0 + x 1 y 1 + x 2 y 2 + x 3 y 3 . {\displaystyle \langle \mathbf {x} |\mathbf {y} \rangle =-x_{0}y_{0}+x_{1}y_{1}+x_{2}y_{2}+x_{3}y_{3}.} Main article: Topological vector space Convergence questions are treated by considering vector spaces V {\displaystyle V} carrying a compatible topology, a structure that allows one to talk about elements being close to each other.[67] Compatible here means that addition and scalar multiplication have to be continuous maps. Roughly, if x {\displaystyle \mathbf {x} } and y {\displaystyle \mathbf {y} } in V {\displaystyle V} , and a {\displaystyle a} in F {\displaystyle F} vary by a bounded amount, then so do x + y {\displaystyle \mathbf {x} +\mathbf {y} } and a x . {\displaystyle a\mathbf {x} .} [nb 6] To make sense of specifying the amount a scalar changes, the field F {\displaystyle F} also has to carry a topology in this context; a common choice is the reals or the complex numbers. In such topological vector spaces one can consider series of vectors. The infinite sum ∑ i = 1 ∞ f i = lim n → ∞ f 1 + ⋯ + f n {\displaystyle \sum _{i=1}^{\infty }f_{i}~=~\lim _{n\to \infty }f_{1}+\cdots +f_{n}} denotes the limit of the corresponding finite partial sums of the sequence f 1 , f 2 , … {\displaystyle f_{1},f_{2},\ldots } of elements of V . {\displaystyle V.} For example, the f i {\displaystyle f_{i}} could be (real or complex) functions belonging to some function space V , {\displaystyle V,} in which case the series is a function series. The mode of convergence of the series depends on the topology imposed on the function space. In such cases, pointwise convergence and uniform convergence are two prominent examples.[68] Unit "spheres" in R 2 {\displaystyle \mathbf {R} ^{2}} consist of plane vectors of norm 1. Depicted are the unit spheres in different p {\displaystyle p} -norms, for p = 1 , 2 , {\displaystyle p=1,2,} and ∞ . {\displaystyle \infty .} The bigger diamond depicts points of 1-norm equal to 2. A way to ensure the existence of limits of certain infinite series is to restrict attention to spaces where any Cauchy sequence has a limit; such a vector space is called complete. Roughly, a vector space is complete provided that it contains all necessary limits. For example, the vector space of polynomials on the unit interval [0 , 1] , {\displaystyle [0,1],} equipped with the topology of uniform convergence is not complete because any continuous function on [0 , 1] {\displaystyle [0,1]} can be uniformly approximated by a sequence of polynomials, by the Weierstrass approximation theorem.[69] In contrast, the space of all continuous functions on [0 , 1] {\displaystyle [0,1]} with the same topology is complete.[70] A norm gives rise to a topology by defining that a sequence of vectors v n {\displaystyle \mathbf {v} _{n}} converges to v {\displaystyle \mathbf {v} } if and only if lim n → ∞ | v n − v | = 0. {\displaystyle \lim _{n\to \infty }|\mathbf {v} _{n}-\mathbf {v} |=0.} Banach and Hilbert spaces are complete topological vector spaces whose topologies are given, respectively, by a norm and an inner product. Their study—a key piece of functional analysis—focuses on infinite-dimensional vector spaces, since all norms on finite-dimensional topological vector spaces give rise to the same notion of convergence.[71] The image at the right shows the equivalence of the 1 {\displaystyle 1} -norm and ∞ {\displaystyle \infty } -norm on R 2 : {\displaystyle \mathbf {R} ^{2}:} as the unit "balls" enclose each other, a sequence converges to zero in one norm if and only if it so does in the other norm. In the infinite-dimensional case, however, there will generally be inequivalent topologies, which makes the study of topological vector spaces richer than that of vector spaces without additional data. From a conceptual point of view, all notions related to topological vector spaces should match the topology. For example, instead of considering all linear maps (also called functionals) V → W , {\displaystyle V\to W,} maps between topological vector spaces are required to be continuous.[72] In particular, the (topological) dual space V ∗ {\displaystyle V^{*}} consists of continuous functionals V → R {\displaystyle V\to \mathbf {R} } (or to C {\displaystyle \mathbf {C} }). The fundamental Hahn–Banach theorem is concerned with separating subspaces of appropriate topological vector spaces by continuous functionals.[73] Main article: Banach space Banach spaces, introduced by Stefan Banach, are complete normed vector spaces.[74] A first example is the vector space ℓ p {\displaystyle \ell ^{p}} consisting of infinite vectors with real entries x = (x 1 , x 2 , … , x n , …) {\displaystyle \mathbf {x} =\left(x_{1},x_{2},\ldots ,x_{n},\ldots \right)} whose p {\displaystyle p} -norm (1 ≤ p ≤ ∞) {\displaystyle (1\leq p\leq \infty)} given by ‖ x ‖ ∞ := sup i | x i | for p = ∞ , and {\displaystyle \|\mathbf {x} \|_{\infty }:=\sup _{i}|x_{i}|\qquad {\text{ for }}p=\infty ,\ {\text{ and }}} ‖ x ‖ p := (∑ i | x i | p) 1 p for p < ∞ . {\displaystyle \|\mathbf {x} \|_{p}:=\left(\sum _{i}|x_{i}|^{p}\right)^{\frac {1}{p}}\qquad {\text{ for }}p0 {\displaystyle r>0} and center (x 0 , y 0 , z 0) {\displaystyle (x_{0},y_{0},z_{0})} can be parameterized using trigonometric functions. x = x 0 + r sin ⁡ θ cos ⁡ φ y = y 0 + r sin ⁡ θ sin ⁡ φ z = z 0 + r cos ⁡ θ {\displaystyle {\begin{aligned}x&=x_{0}+r\sin \theta \;\cos \varphi \\y&=y_{0}+r\sin \theta \;\sin \varphi \\z&=z_{0}+r\cos \theta \,\end{aligned}}} [5] The symbols used here are the same as those used in spherical coordinates. r is constant, while θ varies from 0 to π and φ {\displaystyle \varphi } varies from 0 to 2π. Sphere and circumscribed cylinder In three dimensions, the volume inside a sphere (that is, the volume of a ball, but classically referred to as the volume of a sphere) is V = 4 3 π r 3 = π 6 d 3 ≈ 0.5236 ⋅ d 3 {\displaystyle V={\frac {4}{3}}\pi r^{3}={\frac {\pi }{6}}\ d^{3}\approx 0.5236\cdot d^{3}} where r is the radius and d is the diameter of the sphere. Archimedes first derived this formula (On the Sphere and Cylinder c. 225 BCE) by showing that the volume inside a sphere is twice the volume between the sphere and the circumscribed cylinder of that sphere (having the height and diameter equal to the diameter of the sphere).[6] This may be proved by inscribing a cone upside down into semi-sphere, noting that the area of a cross section of the cone plus the area of a cross section of the sphere is the same as the area of the cross section of the circumscribing cylinder, and applying Cavalieri's principle.[7] This formula can also be derived using integral calculus (i.e., disk integration) to sum the volumes of an infinite number of circular disks of infinitesimally small thickness stacked side by side and centered along the x-axis from x = −r to x = r, assuming the sphere of radius r is centered at the origin. Proof of sphere volume, using calculus At any given x, the incremental volume (δV) equals the product of the cross-sectional area of the disk at x and its thickness (δx): δ V ≈ π y 2 ⋅ δ x . {\displaystyle \delta V\approx \pi y^{2}\cdot \delta x.} The total volume is the summation of all incremental volumes: V ≈ ∑ π y 2 ⋅ δ x . {\displaystyle V\approx \sum \pi y^{2}\cdot \delta x.} In the limit as δx approaches zero,[8] this equation becomes: V = ∫ − r r π y 2 d x . {\displaystyle V=\int _{-r}^{r}\pi y^{2}dx.} At any given x, a right-angled triangle connects x, y and r to the origin; hence, applying the Pythagorean theorem yields: y 2 = r 2 − x 2 . {\displaystyle y^{2}=r^{2}-x^{2}.} Using this substitution gives V = ∫ − r r π (r 2 − x 2) d x , {\displaystyle V=\int _{-r}^{r}\pi \left(r^{2}-x^{2}\right)dx,} which can be evaluated to give the result V = π [r 2 x − x 3 3] − r r = π (r 3 − r 3 3) − π (− r 3 + r 3 3) = 4 3 π r 3 . {\displaystyle V=\pi \left[r^{2}x-{\frac {x^{3}}{3}}\right]_{-r}^{r}=\pi \left(r^{3}-{\frac {r^{3}}{3}}\right)-\pi \left(-r^{3}+{\frac {r^{3}}{3}}\right)={\frac {4}{3}}\pi r^{3}.} An alternative formula is found using spherical coordinates, with volume element d V = r 2 sin ⁡ θ d r d θ d φ {\displaystyle dV=r^{2}\sin \theta \,dr\,d\theta \,d\varphi } so V = ∫ 0 2 π ∫ 0 π ∫ 0 r r ′ 2 sin ⁡ θ d r ′ d θ d φ = 2 π ∫ 0 π ∫ 0 r r ′ 2 sin ⁡ θ d r ′ d θ = 4 π ∫ 0 r r ′ 2 d r ′ = 4 3 π r 3 . {\displaystyle V=\int _{0}^{2\pi }\int _{0}^{\pi }\int _{0}^{r}r'^{2}\sin \theta \,dr'\,d\theta \,d\varphi =2\pi \int _{0}^{\pi }\int _{0}^{r}r'^{2}\sin \theta \,dr'\,d\theta =4\pi \int _{0}^{r}r'^{2}\,dr'\ ={\frac {4}{3}}\pi r^{3}.} For most practical purposes, the volume inside a sphere inscribed in a cube can be approximated as 52.4% of the volume of the cube, since V = π/6 d3, where d is the diameter of the sphere and also the length of a side of the cube and π/6 ≈ 0.5236. For example, a sphere with diameter 1 m has 52.4% the volume of a cube with edge length 1 m, or about 0.524 m3. The surface area of a sphere of radius r is: A = 4 π r 2 . {\displaystyle A=4\pi r^{2}.} Archimedes first derived this formula[9] from the fact that the projection to the lateral surface of a circumscribed cylinder is area-preserving.[10] Another approach to obtaining the formula comes from the fact that it equals the derivative of the formula for the volume with respect to r because the total volume inside a sphere of radius r can be thought of as the summation of the surface area of an infinite number of spherical shells of infinitesimal thickness concentrically stacked inside one another from radius 0 to radius r. At infinitesimal thickness the discrepancy between the inner and outer surface area of any given shell is infinitesimal, and the elemental volume at radius r is simply the product of the surface area at radius r and the infinitesimal thickness. Proof of surface area, using calculus At any given radius r,[note 1] the incremental volume (δV) equals the product of the surface area at radius r (A(r)) and the thickness of a shell (δr): δ V ≈ A (r) ⋅ δ r . {\displaystyle \delta V\approx A(r)\cdot \delta r.} The total volume is the summation of all shell volumes: V ≈ ∑ A (r) ⋅ δ r . {\displaystyle V\approx \sum A(r)\cdot \delta r.} In the limit as δr approaches zero[8] this equation becomes: V = ∫ 0 r A (r) d r . {\displaystyle V=\int _{0}^{r}A(r)\,dr.} Substitute V: 4 3 π r 3 = ∫ 0 r A (r) d r . {\displaystyle {\frac {4}{3}}\pi r^{3}=\int _{0}^{r}A(r)\,dr.} Differentiating both sides of this equation with respect to r yields A as a function of r: 4 π r 2 = A (r) . {\displaystyle 4\pi r^{2}=A(r).} This is generally abbreviated as: A = 4 π r 2 , {\displaystyle A=4\pi r^{2},} where r is now considered to be the fixed radius of the sphere. Alternatively, the area element on the sphere is given in spherical coordinates by dA = r2 sin θ dθ dφ. The total area can thus be obtained by integration: A = ∫ 0 2 π ∫ 0 π r 2 sin ⁡ θ d θ d φ = 4 π r 2 . {\displaystyle A=\int _{0}^{2\pi }\int _{0}^{\pi }r^{2}\sin \theta \,d\theta \,d\varphi =4\pi r^{2}.} The sphere has the smallest surface area of all surfaces that enclose a given volume, and it encloses the largest volume among all closed surfaces with a given surface area.[11] The sphere therefore appears in nature: for example, bubbles and small water drops are roughly spherical because the surface tension locally minimizes surface area. The surface area relative to the mass of a ball is called the specific surface area and can be expressed from the above stated equations as S S A = A V ρ = 3 r ρ {\displaystyle \mathrm {SSA} ={\frac {A}{V\rho }}={\frac {3}{r\rho }}} where ρ is the density (the ratio of mass to volume). A sphere can be constructed as the surface formed by rotating a circle one half revolution about any of its diameters; this is very similar to the traditional definition of a sphere as given in Euclid's Elements. Since a circle is a special type of ellipse, a sphere is a special type of ellipsoid of revolution. Replacing the circle with an ellipse rotated about its major axis, the shape becomes a prolate spheroid; rotated about the minor axis, an oblate spheroid.[12] A sphere is uniquely determined by four points that are not coplanar. More generally, a sphere is uniquely determined by four conditions such as passing through a point, being tangent to a plane, etc.[13] This property is analogous to the property that three non-collinear points determine a unique circle in a plane. Consequently, a sphere is uniquely determined by (that is, passes through) a circle and a point not in the plane of that circle. By examining the common solutions of the equations of two spheres, it can be seen that two spheres intersect in a circle and the plane containing that circle is called the radical plane of the intersecting spheres.[14] Although the radical plane is a real plane, the circle may be imaginary (the spheres have no real point in common) or consist of a single point (the spheres are tangent at that point).[15] The angle between two spheres at a real point of intersection is the dihedral angle determined by the tangent planes to the spheres at that point. Two spheres intersect at the same angle at all points of their circle of intersection.[16] They intersect at right angles (are orthogonal) if and only if the square of the distance between their centers is equal to the sum of the squares of their radii.[4] Main article: Pencil (mathematics) § Pencil of spheres If f(x, y, z) = 0 and g(x, y, z) = 0 are the equations of two distinct spheres then s f (x , y , z) + t g (x , y , z) = 0 {\displaystyle sf(x,y,z)+tg(x,y,z)=0} is also the equation of a sphere for arbitrary values of the parameters s and t. The set of all spheres satisfying this equation is called a pencil of spheres determined by the original two spheres. In this definition a sphere is allowed to be a plane (infinite radius, center at infinity) and if both the original spheres are planes then all the spheres of the pencil are planes, otherwise there is only one plane (the radical plane) in the pencil.[4] A normal vector to a sphere, a normal plane and its normal section. The curvature of the curve of intersection is the sectional curvature. For the sphere each normal section through a given point will be a circle of the same radius: the radius of the sphere. This means that every point on the sphere will be an umbilical point. In their book Geometry and the Imagination, David Hilbert and Stephan Cohn-Vossen describe eleven properties of the sphere and discuss whether these properties uniquely determine the sphere.[17] Several properties hold for the plane, which can be thought of as a sphere with infinite radius. These properties are: The points on the sphere are all the same distance from a fixed point. Also, the ratio of the distance of its points from two fixed points is constant. The first part is the usual definition of the sphere and determines it uniquely. The second part can be easily deduced and follows a similar result of Apollonius of Perga for the circle. This second part also holds for the plane. The contours and plane sections of the sphere are circles. This property defines the sphere uniquely. The sphere has constant width and constant girth. The width of a surface is the distance between pairs of parallel tangent planes. Numerous other closed convex surfaces have constant width, for example the Meissner body. The girth of a surface is the circumference of the boundary of its orthogonal projection on to a plane. Each of these properties implies the other. All points of a sphere are umbilics. At any point on a surface a normal direction is at right angles to the surface because on the sphere these are the lines radiating out from the center of the sphere. The intersection of a plane that contains the normal with the surface will form a curve that is called a normal section, and the curvature of this curve is the normal curvature. For most points on most surfaces, different sections will have different curvatures; the maximum and minimum values of these are called the principal curvatures. Any closed surface will have at least four points called umbilical points. At an umbilic all the sectional curvatures are equal; in particular the principal curvatures are equal. Umbilical points can be thought of as the points where the surface is closely approximated by a sphere. For the sphere the curvatures of all normal sections are equal, so every point is an umbilic. The sphere and plane are the only surfaces with this property. The sphere does not have a surface of centers. For a given normal section exists a circle of curvature that equals the sectional curvature, is tangent to the surface, and the center lines of which lie along on the normal line. For example, the two centers corresponding to the maximum and minimum sectional curvatures are called the focal points, and the set of all such centers forms the focal surface. For most surfaces the focal surface forms two sheets that are each a surface and meet at umbilical points. Several cases are special: * For channel surfaces one sheet forms a curve and the other sheet is a surface * For cones, cylinders, tori and cyclides both sheets form curves. * For the sphere the center of every osculating circle is at the center of the sphere and the focal surface forms a single point. This property is unique to the sphere. All geodesics of the sphere are closed curves. Geodesics are curves on a surface that give the shortest distance between two points. They are a generalization of the concept of a straight line in the plane. For the sphere the geodesics are great circles. Many other surfaces share this property. Of all the solids having a given volume, the sphere is the one with the smallest surface area; of all solids having a given surface area, the sphere is the one having the greatest volume. It follows from isoperimetric inequality. These properties define the sphere uniquely and can be seen in soap bubbles: a soap bubble will enclose a fixed volume, and surface tension minimizes its surface area for that volume. A freely floating soap bubble therefore approximates a sphere (though such external forces as gravity will slightly distort the bubble's shape). It can also be seen in planets and stars where gravity minimizes surface area for large celestial bodies. The sphere has the smallest total mean curvature among all convex solids with a given surface area. The mean curvature is the average of the two principal curvatures, which is constant because the two principal curvatures are constant at all points of the sphere. The sphere has constant mean curvature. The sphere is the only embedded surface that lacks boundary or singularities with constant positive mean curvature. Other such immersed surfaces as minimal surfaces have constant mean curvature. The sphere has constant positive Gaussian curvature. Gaussian curvature is the product of the two principal curvatures. It is an intrinsic property that can be determined by measuring length and angles and is independent of how the surface is embedded in space. Hence, bending a surface will not alter the Gaussian curvature, and other surfaces with constant positive Gaussian curvature can be obtained by cutting a small slit in the sphere and bending it. All these other surfaces would have boundaries, and the sphere is the only surface that lacks a boundary with constant, positive Gaussian curvature. The pseudosphere is an example of a surface with constant negative Gaussian curvature. The sphere is transformed into itself by a three-parameter family of rigid motions. Rotating around any axis a unit sphere at the origin will map the sphere onto itself. Any rotation about a line through the origin can be expressed as a combination of rotations around the three-coordinate axis (see Euler angles). Therefore, a three-parameter family of rotations exists such that each rotation transforms the sphere onto itself; this family is the rotation group SO(3). The plane is the only other surface with a three-parameter family of transformations (translations along the x- and y-axes and rotations around the origin). Circular cylinders are the only surfaces with two-parameter families of rigid motions and the surfaces of revolution and helicoids are the only surfaces with a one-parameter family. Main article: Spherical geometry Great circle on a sphere The basic elements of Euclidean plane geometry are points and lines. On the sphere, points are defined in the usual sense. The analogue of the "line" is the geodesic, which is a great circle; the defining characteristic of a great circle is that the plane containing all its points also passes through the center of the sphere. Measuring by arc length shows that the shortest path between two points lying on the sphere is the shorter segment of the great circle that includes the points. Many theorems from classical geometry hold true for spherical geometry as well, but not all do because the sphere fails to satisfy some of classical geometry's postulates, including the parallel postulate. In spherical trigonometry, angles are defined between great circles. Spherical trigonometry differs from ordinary trigonometry in many respects. For example, the sum of the interior angles of a spherical triangle always exceeds 180 degrees. Also, any two similar spherical triangles are congruent. Any pair of points on a sphere that lie on a straight line through the sphere's center (i.e., the diameter) are called antipodal points – on the sphere, the distance between them is exactly half the length of the circumference.[note 2] Any other (i.e., not antipodal) pair of distinct points on a sphere lie on a unique great circle, segment it into one minor (i.e., shorter) and one major (i.e., longer) arc, and have the minor arc's length be the shortest distance between them on the sphere.[note 3] Spherical geometry is a form of elliptic geometry, which together with hyperbolic geometry makes up non-Euclidean geometry. The sphere is a smooth surface with constant Gaussian curvature at each point equal to 1/r2.[9] As per Gauss's Theorema Egregium, this curvature is independent of the sphere's embedding in 3-dimensional space. Also following from Gauss, a sphere cannot be mapped to a plane while maintaining both areas and angles. Therefore, any map projection introduces some form of distortion. A sphere of radius r has area element d A = r 2 sin ⁡ θ d θ d φ {\displaystyle dA=r^{2}\sin \theta \,d\theta \,d\varphi } . This can be found from the volume element in spherical coordinates with r held constant.[9] A sphere of any radius centered at zero is an integral surface of the following differential form: x d x + y d y + z d z = 0. {\displaystyle x\,dx+y\,dy+z\,dz=0.} This equation reflects that the position vector and tangent plane at a point are always orthogonal to each other. Furthermore, the outward-facing normal vector is equal to the position vector scaled by 1/r. In Riemannian geometry, the filling area conjecture states that the hemisphere is the optimal (least area) isometric filling of the Riemannian circle. Remarkably, it is possible to turn an ordinary sphere inside out in a three-dimensional space with possible self-intersections but without creating any creases, in a process called sphere eversion. The antipodal quotient of the sphere is the surface called the real projective plane, which can also be thought of as the Northern Hemisphere with antipodal points of the equator identified. Plane section of a sphere: one circle Coaxial intersection of a sphere and a cylinder: two circles Main article: Circle of a sphere Circles on the sphere are, like circles in the plane, made up of all points a certain distance from a fixed point on the sphere. The intersection of a sphere and a plane is a circle, a point, or empty.[18] Great circles are the intersection of the sphere with a plane passing through the center of a sphere: others are called small circles. More complicated surfaces may intersect a sphere in circles, too: the intersection of a sphere with a surface of revolution whose axis contains the center of the sphere (are coaxial) consists of circles and/or points if not empty. For example, the diagram to the right shows the intersection of a sphere and a cylinder, which consists of two circles. If the cylinder radius were that of the sphere, the intersection would be a single circle. If the cylinder radius were larger than that of the sphere, the intersection would be empty. Main article: Rhumb line Loxodrome In navigation, a loxodrome or rhumb line is a path whose bearing, the angle between its tangent and due North, is constant. Loxodromes project to straight lines under the Mercator projection. Two special cases are the meridians which are aligned directly North–South and parallels which are aligned directly East–West. For any other bearing, a loxodrome spirals infinitely around each pole. For the Earth modeled as a sphere, or for a general sphere given a spherical coordinate system, such a loxodrome is a kind of spherical spiral.[19] Main article: Clélie Clelia spiral with c = 8 Another kind of spherical spiral is the Clelia curve, for which the longitude (or azimuth) φ {\displaystyle \varphi } and the colatitude (or polar angle) θ {\displaystyle \theta } are in a linear relationship, φ = c θ {\displaystyle \varphi =c\theta } . Clelia curves project to straight lines under the equirectangular projection. Viviani's curve (c = 1 {\displaystyle c=1}) is a special case. Clelia curves approximate the ground track of satellites in polar orbit. Main article: Spherical conic The analog of a conic section on the sphere is a spherical conic, a quartic curve which can be defined in several equivalent ways. The intersection of a sphere with a quadratic cone whose vertex is the sphere center The intersection of a sphere with an elliptic or hyperbolic cylinder whose axis passes through the sphere center The locus of points whose sum or difference of great-circle distances from a pair of foci is a constant Many theorems relating to planar conic sections also extend to spherical conics. General intersection sphere-cylinder If a sphere is intersected by another surface, there may be more complicated spherical curves. Example sphere–cylinder Main article: Sphere–cylinder intersection The intersection of the sphere with equation x 2 + y 2 + z 2 = r 2 {\displaystyle \;x^{2}+y^{2}+z^{2}=r^{2}\;} and the cylinder with equation (y − y 0) 2 + z 2 = a 2 , y 0 ≠ 0 {\displaystyle \;(y-y_{0})^{2}+z^{2}=a^{2},\;y_{0}eq 0\;} is not just one or two circles. It is the solution of the non-linear system of equations x 2 + y 2 + z 2 − r 2 = 0 {\displaystyle x^{2}+y^{2}+z^{2}-r^{2}=0} (y − y 0) 2 + z 2 − a 2 = 0 . {\displaystyle (y-y_{0})^{2}+z^{2}-a^{2}=0\ .} (see implicit curve and the diagram) An ellipsoid is a sphere that has been stretched or compressed in one or more directions. More exactly, it is the image of a sphere under an affine transformation. An ellipsoid bears the same relationship to the sphere that an ellipse does to a circle. Main article: n-sphere Spheres can be generalized to spaces of any number of dimensions. For any natural number n, an n-sphere, often denoted Sn, is the set of points in (n + 1)-dimensional Euclidean space that are at a fixed distance r from a central point of that space, where r is, as before, a positive real number. In particular: S0: a 0-sphere consists of two discrete points, −r and r S1: a 1-sphere is a circle of radius r S2: a 2-sphere is an ordinary sphere S3: a 3-sphere is a sphere in 4-dimensional Euclidean space. Spheres for n > 2 are sometimes called hyperspheres. The n-sphere of unit radius centered at the origin is denoted Sn and is often referred to as "the" n-sphere. The ordinary sphere is a 2-sphere, because it is a 2-dimensional surface which is embedded in 3-dimensional space. In topology, the n-sphere is an example of a compact topological manifold without boundary. A topological sphere need not be smooth; if it is smooth, it need not be diffeomorphic to the Euclidean sphere (an exotic sphere). The sphere is the inverse image of a one-point set under the continuous function ‖x‖, so it is closed; Sn is also bounded, so it is compact by the Heine–Borel theorem. Main article: Metric space More generally, in a metric space (E,d), the sphere of center x and radius r > 0 is the set of points y such that d(x,y) = r. If the center is a distinguished point that is considered to be the origin of E, as in a normed space, it is not mentioned in the definition and notation. The same applies for the radius if it is taken to equal one, as in the case of a unit sphere. Unlike a ball, even a large sphere may be an empty set. For example, in Zn with Euclidean metric, a sphere of radius r is nonempty only if r2 can be written as sum of n squares of integers. An octahedron is a sphere in taxicab geometry, and a cube is a sphere in geometry using the Chebyshev distance. The geometry of the sphere was studied by the Greeks. Euclid's Elements defines the sphere in book XI, discusses various properties of the sphere in book XII, and shows how to inscribe the five regular polyhedra within a sphere in book XIII. Euclid does not include the area and volume of a sphere, only a theorem that the volume of a sphere varies as the third power of its diameter, probably due to Eudoxus of Cnidus. The volume and area formulas were first determined in Archimedes's On the Sphere and Cylinder by the method of exhaustion. Zenodorus was the first to state that, for a given surface area, the sphere is the solid of maximum volume.[3] Archimedes wrote about the problem of dividing a sphere into segments whose volumes are in a given ratio, but did not solve it. A solution by means of the parabola and hyperbola was given by Dionysodorus.[20] A similar problem – to construct a segment equal in volume to a given segment, and in surface to another segment – was solved later by al-Quhi.[3] An image of one of the most accurate human-made spheres, as it refracts the image of Einstein in the background. This sphere was a fused quartz gyroscope for the Gravity Probe B experiment, and differs in shape from a perfect sphere by no more than 40 atoms (less than 10 nm) of thickness. It was announced on 1 July 2008 that Australian scientists had created even more nearly perfect spheres, accurate to 0.3 nm, as part of an international hunt to find a new global standard kilogram.[21] Deck of playing cards illustrating engineering instruments, England, 1702. King of spades: Spheres See also: Ball (mathematics) § Regions Hemisphere Spherical cap Spherical lune Spherical polygon Spherical sector Spherical segment Spherical wedge Spherical zone 3-sphere Affine sphere Alexander horned sphere Celestial spheres Curvature Directional statistics Dyson sphere Gauss map Hand with Reflecting Sphere, M.C. Escher self-portrait drawing illustrating reflection and the optical properties of a mirror sphere Hoberman sphere Homology sphere Homotopy groups of spheres Homotopy sphere Lenart Sphere Napkin ring problem Orb (optics) Pseudosphere Riemann sphere Solid angle Sphere packing Spherical coordinates Spherical cow Spherical helix, tangent indicatrix of a curve of constant precession Spherical polyhedron Sphericity Tennis ball theorem Volume-equivalent radius Zoll sphere ^ r is being considered as a variable in this computation. ^ It does not matter which direction is chosen, the distance is the sphere's radius × π. ^ The distance between two non-distinct points (i.e., a point and itself) on the sphere is zero. ^ σφαῖρα, Henry George Liddell, Robert Scott, A Greek-English Lexicon, on Perseus. ^ a b Albert 2016, p. 54. ^ a b c d e f Chisholm, Hugh, ed. (1911). "Sphere" . Encyclopædia Britannica. Vol. 25 (11th ed.). Cambridge University Press. pp. 647–648. ^ a b c Woods 1961, p. 266. ^ Kreyszig (1972, p. 342). ^ Steinhaus 1969, p. 223. ^ "The volume of a sphere – Math Central". mathcentral.uregina.ca. Retrieved 10 June 2019. ^ a b E.J. Borowski; J.M. Borwein (1989). Collins Dictionary of Mathematics. Collins. pp. 141, 149. ISBN 978-0-00-434347-1. ^ a b c Weisstein, Eric W. "Sphere". MathWorld. ^ Steinhaus 1969, p. 221. ^ Osserman, Robert (1978). "The isoperimetric inequality". Bulletin of the American Mathematical Society. 84 (6): 1187. doi:10.1090/S0002-9904-1978-14553-4. Retrieved 14 December 2019. ^ Albert 2016, p. 60. ^ Albert 2016, p. 55. ^ Albert 2016, p. 57. ^ Woods 1961, p. 267. ^ Albert 2016, p. 58. ^ Hilbert, David; Cohn-Vossen, Stephan (1952). "Eleven properties of the sphere". Geometry and the Imagination (2nd ed.). Chelsea. pp. 215–231. ISBN 978-0-8284-1087-8. {{cite book}}: ISBN / Date incompatibility (help) ^ Weisstein, Eric W. "Spheric section". MathWorld. ^ "Loxodrome". ^ Fried, Michael N. (25 February 2019). "conic sections". Oxford Research Encyclopedia of Classics. doi:10.1093/acrefore/9780199381135.013.8161. ISBN 978-0-19-938113-5. Retrieved 4 November 2022. More significantly, Vitruvius (On Architecture, Vitr. 9.8) associated conical sundials with Dionysodorus (early 2nd century bce), and Dionysodorus, according to Eutocius of Ascalon (c. 480–540 ce), used conic sections to complete a solution for Archimedes' problem of cutting a sphere by a plane so that the ratio of the resulting volumes would be the same as a given ratio. ^ New Scientist | Technology | Roundest objects in the world created. Wikisource has the text of the 1911 Encyclopædia Britannica article "Sphere". Sphere at Wikipedia's sister projects Definitions from WiktionaryMedia from CommonsNews from WikinewsQuotations from WikiquoteTexts from WikisourceTextbooks from WikibooksResources from Wikiversity Albert, Abraham Adrian (2016) [1949], Solid Analytic Geometry, Dover, ISBN 978-0-486-81026-3. Dunham, William (1997). The Mathematical Universe: An Alphabetical Journey Through the Great Proofs, Problems and Personalities. New York: Wiley. pp. 28, 226. Bibcode:1994muaa.book.....D. ISBN 978-0-471-17661-9. Kreyszig, Erwin (1972), Advanced Engineering Mathematics (3rd ed.), New York: Wiley, ISBN 978-0-471-50728-4. Steinhaus, H. (1969), Mathematical Snapshots (Third American ed.), Oxford University Press. Woods, Frederick S. (1961) [1922], Higher Geometry / An Introduction to Advanced Methods in Analytic Geometry, Dover. John C. Polking (15 April 1999). "The Geometry of the Sphere". www.math.csi.cuny.edu. Retrieved 21 January 2022. Mathematica/Uniform Spherical Distribution Surface area of sphere proof Retrieved from " 3Mathematical object This article includes a list of general references, but it lacks sufficient corresponding inline citations. Please help to improve this article by introducing more precise citations. (June 2016) (Learn how and when to remove this message) Stereographic projection of the hypersphere's parallels (red), meridians (blue) and hypermeridians (green). Because this projection is conformal, the curves intersect each other orthogonally (in the yellow points) as in 4D. All curves are circles: the curves that intersect ⟨0,0,0,1⟩ have infinite radius (= straight line). In this picture, the whole 3D space maps the surface of the hypersphere, whereas in the next picture the 3D space contained the shadow of the bulk hypersphere. Direct projection of 3-sphere into 3D space and covered with surface grid, showing structure as stack of 3D spheres (2-spheres) In mathematics, a hypersphere or 3-sphere is a 4-dimensional analogue of a sphere, and is the 3-dimensional n-sphere. In 4-dimensional Euclidean space, it is the set of points equidistant from a fixed central point. The interior of a 3-sphere is a 4-ball. It is called a 3-sphere because topologically, the surface itself is 3-dimensional, even though it is curved into the 4th dimension. For example, when traveling on a 3-sphere, you can go north and south, east and west, or along a 3rd set of cardinal directions. This means that a 3-sphere is an example of a 3-manifold. In coordinates, a 3-sphere with center (C0, C1, C2, C3) and radius r is the set of all points (x0, x1, x2, x3) in real, 4-dimensional space (R4) such that ∑ i = 0 3 (x i − C i) 2 = (x 0 − C 0) 2 + (x 1 − C 1) 2 + (x 2 − C 2) 2 + (x 3 − C 3) 2 = r 2 . {\displaystyle \sum _{i=0}^{3}(x_{i}-C_{i})^{2}=(x_{0}-C_{0})^{2}+(x_{1}-C_{1})^{2}+(x_{2}-C_{2})^{2}+(x_{3}-C_{3})^{2}=r^{2}.} The 3-sphere centered at the origin with radius 1 is called the unit 3-sphere and is usually denoted S3: S 3 = { (x 0 , x 1 , x 2 , x 3) ∈ R 4 : x 0 2 + x 1 2 + x 2 2 + x 3 2 = 1 } . {\displaystyle S^{3}=\left\{(x_{0},x_{1},x_{2},x_{3})\in \mathbb {R} ^{4}:x_{0}^{2}+x_{1}^{2}+x_{2}^{2}+x_{3}^{2}=1\right\}.} It is often convenient to regard R4 as the space with 2 complex dimensions (C2) or the quaternions (H). The unit 3-sphere is then given by S 3 = { (z 1 , z 2) ∈ C 2 : | z 1 | 2 + | z 2 | 2 = 1 } {\displaystyle S^{3}=\left\{(z_{1},z_{2})\in \mathbb {C} ^{2}:|z_{1}|^{2}+|z_{2}|^{2}=1\right\}} or S 3 = { q ∈ H : ‖ q ‖ = 1 } . {\displaystyle S^{3}=\left\{q\in \mathbb {H} :\|q\|=1\right\}.} This description as the quaternions of norm one identifies the 3-sphere with the versors in the quaternion division ring. Just as the unit circle is important for planar polar coordinates, so the 3-sphere is important in the polar view of 4-space involved in quaternion multiplication. See polar decomposition of a quaternion for details of this development of the three-sphere. This view of the 3-sphere is the basis for the study of elliptic space as developed by Georges Lemaître.[1] The 3-dimensional surface volume of a 3-sphere of radius r is S V = 2 π 2 r 3 {\displaystyle SV=2\pi ^{2}r^{3}\,} while the 4-dimensional hypervolume (the content of the 4-dimensional region, or ball, bounded by the 3-sphere) is H = 1 2 π 2 r 4 . {\displaystyle H={\frac {1}{2}}\pi ^{2}r^{4}.} Every non-empty intersection of a 3-sphere with a three-dimensional hyperplane is a 2-sphere (unless the hyperplane is tangent to the 3-sphere, in which case the intersection is a single point). As a 3-sphere moves through a given three-dimensional hyperplane, the intersection starts out as a point, then becomes a growing 2-sphere that reaches its maximal size when the hyperplane cuts right through the "equator" of the 3-sphere. Then the 2-sphere shrinks again down to a single point as the 3-sphere leaves the hyperplane. In a given three-dimensional hyperplane, a 3-sphere can rotate about an "equatorial plane" (analogous to a 2-sphere rotating about a central axis), in which case it appears to be a 2-sphere whose size is constant. A 3-sphere is a compact, connected, 3-dimensional manifold without boundary. It is also simply connected. What this means, in the broad sense, is that any loop, or circular path, on the 3-sphere can be continuously shrunk to a point without leaving the 3-sphere. The Poincaré conjecture, proved in 2003 by Grigori Perelman, provides that the 3-sphere is the only three-dimensional manifold (up to homeomorphism) with these properties. The 3-sphere is homeomorphic to the one-point compactification of R3. In general, any topological space that is homeomorphic to the 3-sphere is called a topological 3-sphere. The homology groups of the 3-sphere are as follows: H0(S3, Z) and H3(S3, Z) are both infinite cyclic, while Hi(S3, Z) = {} for all other indices i. Any topological space with these homology groups is known as a homology 3-sphere. Initially Poincaré conjectured that all homology 3-spheres are homeomorphic to S3, but then he himself constructed a non-homeomorphic one, now known as the Poincaré homology sphere. Infinitely many homology spheres are now known to exist. For example, a Dehn filling with slope 1/n on any knot in the 3-sphere gives a homology sphere; typically these are not homeomorphic to the 3-sphere. As to the homotopy groups, we have π1(S3) = π2(S3) = {} and π3(S3) is infinite cyclic. The higher-homotopy groups (k ≥ 4) are all finite abelian but otherwise follow no discernible pattern. For more discussion see homotopy groups of spheres. Homotopy groups of S3 k 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 πk(S3) 0 0 0 Z Z2 Z2 Z12 Z2 Z2 Z3 Z15 Z2 Z2⊕Z2 Z12⊕Z2 Z84⊕Z2⊕Z2 Z2⊕Z2 Z6 The 3-sphere is naturally a smooth manifold, in fact, a closed embedded submanifold of R4. The Euclidean metric on R4 induces a metric on the 3-sphere giving it the structure of a Riemannian manifold. As with all spheres, the 3-sphere has constant positive sectional curvature equal to 1/r2 where r is the radius. Much of the interesting geometry of the 3-sphere stems from the fact that the 3-sphere has a natural Lie group structure given by quaternion multiplication (see the section below on group structure). The only other spheres with such a structure are the 0-sphere and the 1-sphere (see circle group). Unlike the 2-sphere, the 3-sphere admits nonvanishing vector fields (sections of its tangent bundle). One can even find three linearly independent and nonvanishing vector fields. These may be taken to be any left-invariant vector fields forming a basis for the Lie algebra of the 3-sphere. This implies that the 3-sphere is parallelizable. It follows that the tangent bundle of the 3-sphere is trivial. For a general discussion of the number of linear independent vector fields on a n-sphere, see the article vector fields on spheres. There is an interesting action of the circle group T on S3 giving the 3-sphere the structure of a principal circle bundle known as the Hopf bundle. If one thinks of S3 as a subset of C2, the action is given by (z 1 , z 2) ⋅ λ = (z 1 λ , z 2 λ) ∀ λ ∈ T {\displaystyle (z_{1},z_{2})\cdot \lambda =(z_{1}\lambda ,z_{2}\lambda)\quad \forall \lambda \in \mathbb {T} } . The orbit space of this action is homeomorphic to the two-sphere S2. Since S3 is not homeomorphic to S2 × S1, the Hopf bundle is nontrivial. There are several well-known constructions of the three-sphere. Here we describe gluing a pair of three-balls and then the one-point compactification. A 3-sphere can be constructed topologically by "gluing" together the boundaries of a pair of 3-balls. The boundary of a 3-ball is a 2-sphere, and these two 2-spheres are to be identified. That is, imagine a pair of 3-balls of the same size, then superpose them so that their 2-spherical boundaries match, and let matching pairs of points on the pair of 2-spheres be identically equivalent to each other. In analogy with the case of the 2-sphere (see below), the gluing surface is called an equatorial sphere. Note that the interiors of the 3-balls are not glued to each other. One way to think of the fourth dimension is as a continuous real-valued function of the 3-dimensional coordinates of the 3-ball, perhaps considered to be "temperature". We take the "temperature" to be zero along the gluing 2-sphere and let one of the 3-balls be "hot" and let the other 3-ball be "cold". The "hot" 3-ball could be thought of as the "upper hemisphere" and the "cold" 3-ball could be thought of as the "lower hemisphere". The temperature is highest/lowest at the centers of the two 3-balls. This construction is analogous to a construction of a 2-sphere, performed by gluing the boundaries of a pair of disks. A disk is a 2-ball, and the boundary of a disk is a circle (a 1-sphere). Let a pair of disks be of the same diameter. Superpose them and glue corresponding points on their boundaries. Again one may think of the third dimension as temperature. Likewise, we may inflate the 2-sphere, moving the pair of disks to become the northern and southern hemispheres. After removing a single point from the 2-sphere, what remains is homeomorphic to the Euclidean plane. In the same way, removing a single point from the 3-sphere yields three-dimensional space. An extremely useful way to see this is via stereographic projection. We first describe the lower-dimensional version. Rest the south pole of a unit 2-sphere on the xy-plane in three-space. We map a point P of the sphere (minus the north pole N) to the plane by sending P to the intersection of the line NP with the plane. Stereographic projection of a 3-sphere (again removing the north pole) maps to three-space in the same manner. (Notice that, since stereographic projection is conformal, round spheres are sent to round spheres or to planes.) A somewhat different way to think of the one-point compactification is via the exponential map. Returning to our picture of the unit two-sphere sitting on the Euclidean plane: Consider a geodesic in the plane, based at the origin, and map this to a geodesic in the two-sphere of the same length, based at the south pole. Under this map all points of the circle of radius π are sent to the north pole. Since the open unit disk is homeomorphic to the Euclidean plane, this is again a one-point compactification. The exponential map for 3-sphere is similarly constructed; it may also be discussed using the fact that the 3-sphere is the Lie group of unit quaternions. The four Euclidean coordinates for S3 are redundant since they are subject to the condition that x02 + x12 + x22 + x32 = 1. As a 3-dimensional manifold one should be able to parameterize S3 by three coordinates, just as one can parameterize the 2-sphere using two coordinates (such as latitude and longitude). Due to the nontrivial topology of S3 it is impossible to find a single set of coordinates that cover the entire space. Just as on the 2-sphere, one must use at least two coordinate charts. Some different choices of coordinates are given below. It is convenient to have some sort of hyperspherical coordinates on S3 in analogy to the usual spherical coordinates on S2. One such choice — by no means unique — is to use (ψ, θ, φ), where x 0 = r cos ⁡ ψ x 1 = r sin ⁡ ψ cos ⁡ θ x 2 = r sin ⁡ ψ sin ⁡ θ cos ⁡ φ x 3 = r sin ⁡ ψ sin ⁡ θ sin ⁡ φ {\displaystyle {\begin{aligned}x_{0}&=r\cos \psi \\x_{1}&=r\sin \psi \cos \theta \\x_{2}&=r\sin \psi \sin \theta \cos \varphi \\x_{3}&=r\sin \psi \sin \theta \sin \varphi \end{aligned}}} where ψ and θ run over the range 0 to π, and φ runs over 0 to 2π. Note that, for any fixed value of ψ, θ and φ parameterize a 2-sphere of radius r sin ⁡ ψ {\displaystyle r\sin \psi } , except for the degenerate cases, when ψ equals 0 or π, in which case they describe a point. The round metric on the 3-sphere in these coordinates is given by[2] d s 2 = r 2 [d ψ 2 + sin 2 ⁡ ψ (d θ 2 + sin 2 ⁡ θ d φ 2)] {\displaystyle ds^{2}=r^{2}\left[d\psi ^{2}+\sin ^{2}\psi \left(d\theta ^{2}+\sin ^{2}\theta \,d\varphi ^{2}\right)\right]} and the volume form by d V = r 3 (sin 2 ⁡ ψ sin ⁡ θ) d ψ ∧ d θ ∧ d φ . {\displaystyle dV=r^{3}\left(\sin ^{2}\psi \,\sin \theta \right)\,d\psi \wedge d\theta \wedge d\varphi .} These coordinates have an elegant description in terms of quaternions. Any unit quaternion q can be written as a versor: q = e τ ψ = cos ⁡ ψ + τ sin ⁡ ψ {\displaystyle q=e^{\tau \psi }=\cos \psi +\tau \sin \psi } where τ is a unit imaginary quaternion; that is, a quaternion that satisfies τ2 = −1. This is the quaternionic analogue of Euler's formula. Now the unit imaginary quaternions all lie on the unit 2-sphere in Im H so any such τ can be written: τ = (cos ⁡ θ) i + (sin ⁡ θ cos ⁡ φ) j + (sin ⁡ θ sin ⁡ φ) k {\displaystyle \tau =(\cos \theta)i+(\sin \theta \cos \varphi)j+(\sin \theta \sin \varphi)k} With τ in this form, the unit quaternion q is given by q = e τ ψ = x 0 + x 1 i + x 2 j + x 3 k {\displaystyle q=e^{\tau \psi }=x_{0}+x_{1}i+x_{2}j+x_{3}k} where x0,1,2,3 are as above. When q is used to describe spatial rotations (cf. quaternions and spatial rotations), it describes a rotation about τ through an angle of 2ψ. The Hopf fibration can be visualized using a stereographic projection of S3 to R3 and then compressing R3 to a ball. This image shows points on S2 and their corresponding fibers with the same color. For unit radius another choice of hyperspherical coordinates, (η, ξ1, ξ2), makes use of the embedding of S3 in C2. In complex coordinates (z1, z2) ∈ C2 we write z 1 = e i ξ 1 sin ⁡ η z 2 = e i ξ 2 cos ⁡ η . {\displaystyle {\begin{aligned}z_{1}&=e^{i\,\xi _{1}}\sin \eta \\z_{2}&=e^{i\,\xi _{2}}\cos \eta .\end{aligned}}} This could also be expressed in R4 as x 0 = cos ⁡ ξ 1 sin ⁡ η x 1 = sin ⁡ ξ 1 sin ⁡ η x 2 = cos ⁡ ξ 2 cos ⁡ η x 3 = sin ⁡ ξ 2 cos ⁡ η . {\displaystyle {\begin{aligned}x_{0}&=\cos \xi _{1}\sin \eta \\x_{1}&=\sin \xi _{1}\sin \eta \\x_{2}&=\cos \xi _{2}\cos \eta \\x_{3}&=\sin \xi _{2}\cos \eta .\end{aligned}}} Here η runs over the range 0 to π/2, and ξ1 and ξ2 can take any values between 0 and 2π. These coordinates are useful in the description of the 3-sphere as the Hopf bundle S 1 → S 3 → S 2 . {\displaystyle S^{1}\to S^{3}\to S^{2}.\,} A diagram depicting the poloidal (ξ1) direction, represented by the red arrow, and the toroidal (ξ2) direction, represented by the blue arrow, although the terms poloidal and toroidal are arbitrary in this flat torus case. For any fixed value of η between 0 and π/2, the coordinates (ξ1, ξ2) parameterize a 2-dimensional torus. Rings of constant ξ1 and ξ2 above form simple orthogonal grids on the tori. See image to right. In the degenerate cases, when η equals 0 or π/2, these coordinates describe a circle. The round metric on the 3-sphere in these coordinates is given by d s 2 = d η 2 + sin 2 ⁡ η d ξ 1 2 + cos 2 ⁡ η d ξ 2 2 {\displaystyle ds^{2}=d\eta ^{2}+\sin ^{2}\eta \,d\xi _{1}^{2}+\cos ^{2}\eta \,d\xi _{2}^{2}} and the volume form by d V = sin ⁡ η cos ⁡ η d η ∧ d ξ 1 ∧ d ξ 2 . {\displaystyle dV=\sin \eta \cos \eta \,d\eta \wedge d\xi _{1}\wedge d\xi _{2}.} To get the interlocking circles of the Hopf fibration, make a simple substitution in the equations above[3] z 1 = e i (ξ 1 + ξ 2) sin ⁡ η z 2 = e i (ξ 2 − ξ 1) cos ⁡ η . {\displaystyle {\begin{aligned}z_{1}&=e^{i\,(\xi _{1}+\xi _{2})}\sin \eta \\z_{2}&=e^{i\,(\xi _{2}-\xi _{1})}\cos \eta .\end{aligned}}} In this case η, and ξ1 specify which circle, and ξ2 specifies the position along each circle. One round trip (0 to 2π) of ξ1 or ξ2 equates to a round trip of the torus in the 2 respective directions. Another convenient set of coordinates can be obtained via stereographic projection of S3 from a pole onto the corresponding equatorial R3 hyperplane. For example, if we project from the point (−1, 0, 0, 0) we can write a point p in S3 as p = (1 − ‖ u ‖ 2 1 + ‖ u ‖ 2 , 2 u 1 + ‖ u ‖ 2) = 1 + u 1 − u {\displaystyle p=\left({\frac {1-\|u\|^{2}}{1+\|u\|^{2}}},{\frac {2\mathbf {u} }{1+\|u\|^{2}}}\right)={\frac {1+\mathbf {u} }{1-\mathbf {u} }}} where u = (u1, u2, u3) is a vector in R3 and ‖u‖2 = u12 + u22 + u32. In the second equality above, we have identified p with a unit quaternion and u = u1i + u2j + u3k with a pure quaternion. (Note that the numerator and denominator commute here even though quaternionic multiplication is generally noncommutative). The inverse of this map takes p = (x0, x1, x2, x3) in S3 to u = 1 1 + x 0 (x 1 , x 2 , x 3) . {\displaystyle \mathbf {u} ={\frac {1}{1+x_{0}}}\left(x_{1},x_{2},x_{3}\right).} We could just as well have projected from the point (1, 0, 0, 0), in which case the point p is given by p = (− 1 + ‖ v ‖ 2 1 + ‖ v ‖ 2 , 2 v 1 + ‖ v ‖ 2) = − 1 + v 1 + v {\displaystyle p=\left({\frac {-1+\|v\|^{2}}{1+\|v\|^{2}}},{\frac {2\mathbf {v} }{1+\|v\|^{2}}}\right)={\frac {-1+\mathbf {v} }{1+\mathbf {v} }}} where v = (v1, v2, v3) is another vector in R3. The inverse of this map takes p to v = 1 1 − x 0 (x 1 , x 2 , x 3) . {\displaystyle \mathbf {v} ={\frac {1}{1-x_{0}}}\left(x_{1},x_{2},x_{3}\right).} Note that the u coordinates are defined everywhere but (−1, 0, 0, 0) and the v coordinates everywhere but (1, 0, 0, 0). This defines an atlas on S3 consisting of two coordinate charts or "patches", which together cover all of S3. Note that the transition function between these two charts on their overlap is given by v = 1 ‖ u ‖ 2 u {\displaystyle \mathbf {v} ={\frac {1}{\|u\|^{2}}}\mathbf {u} } and vice versa. When considered as the set of unit quaternions, S3 inherits an important structure, namely that of quaternionic multiplication. Because the set of unit quaternions is closed under multiplication, S3 takes on the structure of a group. Moreover, since quaternionic multiplication is smooth, S3 can be regarded as a real Lie group. It is a nonabelian, compact Lie group of dimension 3. When thought of as a Lie group, S3 is often denoted Sp(1) or U(1, H). It turns out that the only spheres that admit a Lie group structure are S1, thought of as the set of unit complex numbers, and S3, the set of unit quaternions (The degenerate case S0 which consists of the real numbers 1 and −1 is also a Lie group, albeit a 0-dimensional one). One might think that S7, the set of unit octonions, would form a Lie group, but this fails since octonion multiplication is nonassociative. The octonionic structure does give S7 one important property: parallelizability. It turns out that the only spheres that are parallelizable are S1, S3, and S7. By using a matrix representation of the quaternions, H, one obtains a matrix representation of S3. One convenient choice is given by the Pauli matrices: x 1 + x 2 i + x 3 j + x 4 k ↦ (x 1 + i x 2 x 3 + i x 4 − x 3 + i x 4 x 1 − i x 2) . {\displaystyle x_{1}+x_{2}i+x_{3}j+x_{4}k\mapsto {\begin{pmatrix}\;\;\,x_{1}+ix_{2}&x_{3}+ix_{4}\\-x_{3}+ix_{4}&x_{1}-ix_{2}\end{pmatrix}}.} This map gives an injective algebra homomorphism from H to the set of 2 × 2 complex matrices. It has the property that the absolute value of a quaternion q is equal to the square root of the determinant of the matrix image of q. The set of unit quaternions is then given by matrices of the above form with unit determinant. This matrix subgroup is precisely the special unitary group SU(2). Thus, S3 as a Lie group is isomorphic to SU(2). Using our Hopf coordinates (η, ξ1, ξ2) we can then write any element of SU(2) in the form (e i ξ 1 sin ⁡ η e i ξ 2 cos ⁡ η − e − i ξ 2 cos ⁡ η e − i ξ 1 sin ⁡ η) . {\displaystyle {\begin{pmatrix}e^{i\,\xi _{1}}\sin \eta &e^{i\,\xi _{2}}\cos \eta \\-e^{-i\,\xi _{2}}\cos \eta &e^{-i\,\xi _{1}}\sin \eta \end{pmatrix}}.} Another way to state this result is if we express the matrix representation of an element of SU(2) as an exponential of a linear combination of the Pauli matrices. It is seen that an arbitrary element U ∈ SU(2) can be written as U = exp ⁡ (∑ i = 1 3 α i J i) . {\displaystyle U=\exp \left(\sum _{i=1}^{3}\alpha _{i}J_{i}\right).} [4] The condition that the determinant of U is +1 implies that the coefficients α1 are constrained to lie on a 3-sphere. In Edwin Abbott Abbott's Flatland, published in 1884, and in Sphereland, a 1965 sequel to Flatland by Dionys Burger, the 3-sphere is referred to as an oversphere, and a 4-sphere is referred to as a hypersphere. Writing in the American Journal of Physics,[5] Mark A. Peterson describes three different ways of visualizing 3-spheres and points out language in The Divine Comedy that suggests Dante viewed the Universe in the same way; Carlo Rovelli supports the same idea.[6] In Art Meets Mathematics in the Fourth Dimension,[7] Stephen L. Lipscomb develops the concept of the hypersphere dimensions as it relates to art, architecture, and mathematics. 1-sphere, 2-sphere, n-sphere tesseract, polychoron, simplex Pauli matrices Hopf bundle, Riemann sphere Poincaré sphere Reeb foliation Clifford torus ^ Lemaître, Georges (1948). "Quaternions et espace elliptique". Acta. 12. Pontifical Academy of Sciences: 57–78. ^ Landau, Lev D.; Lifshitz, Evgeny M. (1988). Classical Theory of Fields. Course of Theoretical Physics. Vol. 2 (7th ed.). Moscow: Nauka. p. 385. ISBN 978-5-02-014420-0. ^ Banchoff, Thomas. "The Flat Torus in the Three-Sphere". ^ Schwichtenberg, Jakob (2015). Physics from symmetry. Cham: Springer. ISBN 978-3-319-19201-7. OCLC 910917227. ^ Peterson, Mark A. (1979). "Dante and the 3-sphere". American Journal of Physics. 47 (12): 1031–1035. Bibcode:1979AmJPh..47.1031P. doi:10.1119/1.11968. Archived from the original on 23 February 2013. ^ Rovelli, Carlo (9 September 2021). General Relativity: The Essentials. Cambridge: Cambridge University Press. ISBN 978-1-00-901369-7. Retrieved 13 September 2021. ^ Lipscomb, Stephen (2014). Art meets mathematics in the fourth dimension (2 ed.). Berlin: Springer. ISBN 978-3-319-06254-9. OCLC 893872366. Henderson, David W. (2001). "Chapter 20: 3-spheres and hyperbolic 3-spaces". Experiencing Geometry: In Euclidean, Spherical, and Hyperbolic Spaces (second ed.). Prentice-Hall. Archived from the original on 2018-06-19. Weeks, Jeffrey R. (1985). "Chapter 14: The Hypersphere". The Shape of Space: How to Visualize Surfaces and Three-dimensional Manifolds. A Warning on terminology: Our two-sphere is defined in three-dimensional space, where it is the boundary of a three-dimensional ball. This terminology is standard among mathematicians, but not among physicists. So don't be surprised if you find people calling the two-sphere a three-sphere. Zamboj, Michal (8 Jan 2021). "Synthetic construction of the Hopf fibration in a double orthogonal projection of 4-space". Journal of Computational Design and Engineering. 8 (3): 836–854. arXiv:2003.09236v2. doi:10.1093/jcde/qwab018. Weisstein, Eric W. "Hypersphere". MathWorld. Note: This article uses the alternate naming scheme for spheres in which a sphere in n-dimensional space is termed an n-sphere. Retrieved from " 4 The following pages link to 3-sphere External tools (link count transclusion count sorted list) · See help page for transcluding these entries Showing 50 items. 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