I'm not a robot

For real matrices, $\Ran A \neq \Ker A^T$. This can be shown by considering the rank-nullity theorem, which states that for any matrix A, $\operatorname{rank}(A) + nul(A) = D$, where D is the dimension of the domain. If we assume that $\Ran A = \Ker A^T$, then we would have $\operatorname{rank}(A) = \dim(\Ker A)$ and $\operatorname{rank}(A^T) = \dim(\Ker A)$, which implies that $\operatorname{rank}(A) + \operatorname{rank}(A^T) = 2\dim(\Ker A)$. However, since the rank of a matrix is equal to the dimension of its column space and the rank of A^T is equal to the dimension of its row space, we have $\operatorname{rank}(A) + \operatorname{rank}(A^T) = n$, which is greater than or equal to $2\dim(\Ker A)$. Therefore, for real matrices, it is not possible to have $\Ran A = \Ker A^T$. For complex matrices, the situation is different. If we consider a symmetric matrix A, then we can show that $\Ran A = \Ker A^T$ by examining the properties of the orthogonal complement. Note that for any matrix B, ${\bf Ker}B^{\bf T} \supseteq {\bf Im}A^{\bf T}$ is true, but it's not necessarily the other way around. However, if we have ${\bf Ker}B^{\bf T} = {\bf Im}A^{\bf T}$ for some matrix B, then this implies that B and A are related in a specific way. Regarding the existence of a 2×2 matrix A such that $\operatorname{im}(A) = \ker(A)$, we can show that such a matrix exists. Consider the matrix $$A=\begin{pmatrix}1&0\\0&0\end{pmatrix}$$. Then, we have $\operatorname{im}(A) = span(\begin{pmatrix}1\\0\end{pmatrix})$ and $\ker(A) = \{x \in \mathbb{R}^2 | A x = 0\} = span(\begin{pmatrix}0\\1\end{pmatrix})$. Therefore, we have $\operatorname{im}(A) = \ker(A)$. Finally, let's consider the example you provided. For the matrix $$A=\begin{pmatrix}1&i\\i&-1\end{pmatrix}$$ we can show that $\Ker A = \Ran A^T$. The rows of A are $(1,i)$ and $(i,-1)$. If we set $c_1 = 1$ and $c_2 = -i$, then we have $$\sum_i c_i A_i = (1)(1) + (-i)(i) = 1$$ This shows that the right-hand side is contained in the left. Conversely, if $w \in \mathbb{C}^2$ is such that $w \cdot \sum_i c_i A_i = 0$, then we must have $(c_1,c_2) w = 0$. However, since the matrix is symmetric, this implies that $\operatorname{Ker}A=\operatorname{Ker}A^T=\operatorname{Im}(A)$. We can now determine the condition for this to happen. This occurs when the matrix is symmetric, i.e., $A = A^T$. Note that these results don't hold in general. For example, if we consider the matrix $$B=\begin{pmatrix}1&0\\0&0\end{pmatrix}$$ then we have ${\bf Ker}B^{\bf T}= {\bf Im}(A)$ but ${\bf Ker}B^{\bf T}\neq {\bf Im}A$. ###ARTICLELet A and B be two commutative square matrices over a commutative field such that $\operatorname{Im}(A)=\ker(A)$ and $\operatorname{Im}(B)=\ker(B)$. We need to prove that $\ker(AB) = \ker(A) + \ker(B)$. First, we note that if $\operatorname{Im}(A)=\ker(A)$, then $A^2=0$, which implies that $\ker(A)\subset\ker(A+I)$. This is because for any vector x, $Ax=0$ and $(A+I)x=x \Rightarrow x\in\ker(A+I)$. Similarly, if $\operatorname{Im}(B)=\ker(B)$, then $B^2=0$ which implies that $\ker(B)\subset\ker(B+I)$. Again for any vector x, $Bx=0$ and $(B+I)x=x \Rightarrow x\in\ker(B+I)$. Now, we know that if P,Q are orthogonal projections, then $Q-P$ is a projection as well. This is easy to see since for any vector y on the range of Q, we have $(Q-P)y=0$ and similarly, we can show this holds when $Q-P=y$. Let's set $P=A+A^T$ and $Q=B+B^T$. Then we know that P and Q are orthogonal projections onto $\operatorname{Im}(A)$ and $\operatorname{Im}(B)$ respectively. We want to show that $\ker(P+Q) = \ker(A)+\ker(B)$. Notice that if $y\in\ker(P+Q)$, then we must have $(P+Q)y=0$ which means $Py=-Qy$, hence either $y=Py$ or $y=Qy$. Since P and Q are projections, we know the latter holds for some subspace of $\mathbb{R}^n$, but that is clearly outside of our desired solution. This implies that we can only conclude $y\in Py=\ker(A)$ or $y\in Qy=\ker(B)$. Hence, we have shown $\ker(P+Q)\subset \ker(A)+\ker(B)$. The reverse inclusion follows from the obvious fact that any vector in $\ker(A)+\ker(B)$ will certainly be a linear combination of vectors in each kernel. We can also show this works using the properties of orthogonal projections and the characteristic polynomial of the matrices.