By the end of this section, you will be able to: Derive the equation for rotational work. Calculate rotational kinetic energy. Demonstrate the Law of Conservation of Energy. In this module, we will learn about work and energy associated with rotational motion. Figure 1 shows a worker using an electric grindstone propelled by a motor. Sparks are flying, and noise and vibration are created as layers of steel are pared from the pole. The stone continues to turn even after the motor is turned off, but it is eventually brought to a stop by friction. Clearly, the motor had to work to get the stone spinning. This work went into heat, light, sound, vibration, and considerable rotational kinetic energy. Figure 1. The motor works in spinning the grindstone, giving it rotational kinetic energy. That energy is then converted to heat, light, sound, and vibration. (credit: U.S. Navy photo by Mass Communication Specialist Seaman Zachary David Bell) Work must be done to rotate objects such as grindstones or merry-go-rounds. Work was defined in Uniform Circular Motion and Gravitation for translational motion, and we can build on that knowledge when considering work done in rotational motion. The simplest rotational situation is one in which the net force is exerted perpendicular to the radius of a disk (as shown in Figure 2) and remains perpendicular as the disk starts to rotate. The force is parallel to the displacement, and so the net work done is the product of the force times the arc length traveled: [latex]\text{net }W= (\text{net }F)\Delta{s}[/latex]. To get torque and other rotational quantities into the equation, we multiply and divide the right-hand side of the equation by r, and gather terms: [latex]\text{net}W=\left(r\text{ net }F\right)\frac{\Delta s}{r}[/latex]. We recognize that r net F = net F τ and Δs/r = θ, so that [latex]\text{net }W=(\text{net }\tau)\theta[/latex]. This equation is the expression for rotational work. It is very similar to the familiar definition of translational work as force multiplied by distance. Here, torque is analogous to force, and angle is analogous to distance. The equation net W = (net τ)θ is valid in general, even though it was derived for a special case. To get an expression for rotational kinetic energy, we must again perform some algebraic manipulations. The first step is to note that net τ = Iα, so that net W = Iαθ Figure 2. The net force on the disk is kept perpendicular to its radius as the force causes the disk to rotate. The net work done is thus (net F)Δs. The net work goes into rotational kinetic energy. Now, we solve one of the rotational kinematics equations for αθ. We start with the equation ω2 = ωo2 + 2αθ. Next, we solve for αθ: [latex]\alpha\theta=\frac{{\omega}^{2}-{{\omega}_{\text{0}}}^{2}}{2}[/latex]. Substituting this into the equation for net W and gathering terms yields [latex]\text{net }W=\frac{1}{2}I\omega^{2}-\frac{1}{2}{I\omega_{\text{0}}}^{2}[/latex]. This equation is the work-energy theorem for rotational motion only. As you may recall, net work changes the kinetic energy of a system. Through an analogy with translational motion, we define the term [latex]\left(\frac{1}{2}\right){\mathrm{I\omega }}^{2}[/latex] to be rotational kinetic energy KErot for an object with a moment of inertia I and an angular velocity ω: [latex]{\text{KE}}_{\text{rot}}=\frac{1}{2}{{I\omega }}^{2}[/latex]. The expression for rotational kinetic energy is exactly analogous to translational kinetic energy, with I being analogous to m and ω to v. Rotational kinetic energy has important effects. Flywheels, for example, can be used to store large amounts of rotational kinetic energy in a vehicle, as seen in Figure 3. Figure 3. Experimental vehicles, such as this bus in Figure 10.16, have been constructed in which rotational kinetic energy is stored in a large flywheel. When the bus goes down a hill, its transmission converts its gravitational potential energy into KErot. It can also convert translational kinetic energy, when the bus stops, into KErot. The flywheel's energy can then be used to accelerate, to go up another hill, or to keep the bus from going against friction. Consider a person who spins a large grindstone by placing her hand on its edge and exerting a force through part of a revolution as shown in Figure 4. In this example, we verify that the work done by the torque she exerts equals the change in rotational energy. (a) How much work is done if she exerts a force of 200 N through a rotation of 1.00 rad (57.3º)? The force is kept perpendicular to the grindstone's 0.320-m radius at the point of application, and the effects of friction are negligible. (b) What is the final angular velocity if the grindstone has a mass of 85.0 kg? (c) What is the final rotational kinetic energy? (It should equal the work.) Strategy To find the work, we can use the equation net W = (net τ)θ. We have enough information to calculate the torque and are given the rotation angle. In the second part, we can find the final angular velocity using one of the kinematic relationships. In the last part, we can calculate the rotational kinetic energy from its expression in [latex]{\text{KE}}_{\text{rot}}=\frac{1}{2}{\mathrm{I\omega }}^{2}[/latex]. Solution for (a) The net work is expressed in the equation [latex]\text{net }W=(\text{net }\tau)\theta[/latex]. where net τ is the applied force multiplied by the radius (rF) because there is no retarding friction, and the force is perpendicular to r. The angle θ is given. Substituting the given values in the equation above yields [latex]\begin{array}{lll}\text{net }W& =& rF\theta =\left(\text{0.320 m}\right)\left(\text{200 N}\right)\left(\text{1.00 rad}\right)\\ & =& \text{64.0 N}\cdot text{m.}\end{array}[/latex] Noting that 1 N·m = 1 J, net W = 64.0 J Figure 4. A large grindstone is given a spin by a person grasping its outer edge. Solution for (b) To find ω from the given information requires more than one step. We start with the kinematic relationship in the equation Note that ω0 = 0 because we start from rest. Taking the square root of the resulting equation gives ω2 = ω02+2αθ. [latex]\omega =(2\alpha\theta)^{1/2}[/latex] Now we need to find α. One possibility is [latex]\alpha =\frac{\text{net }\tau}{I}[/latex], where the torque is net τ = rF = (0.320 m)(200 N) = 64.0 N · m. The formula for the moment of inertia for a disk is found in Figure 5: Figure 5. Some rotational inertias. [latex]I=\frac{1}{2}{\text{MR}}^{2}=0.5\left(\text{85.0 kg}\right){\left(\text{0.320 m}\right)}^{2}=\text{4.352 kg}\cdot {\text{m}}^{2}[/latex]. Substituting the values of torque and moment of inertia into the expression for α, we obtain [latex]\alpha =\frac{\text{64}\text{.}\text{0 N}\cdot \text{m}}{\text{4.352 kg}\cdot {\text{m}}^{2}}=\text{14.7}\frac{\text{rad}}{{\text{s}}^{2}}[/latex]. Now, substitute this value and the given value for θ into the above expression for ω: [latex]\omega = (2\alpha\theta)^{1/2}={\left[2\left(\text{14.7}\frac{\text{rad}}{{\text{s}}^{2}}\right)\left(\text{1.00 rad}\right)\right]}^{1/2}=\text{5.42}\frac{\text{rad}}{\text{s}}[/latex]. Solution for (c) The final rotational kinetic energy is [latex]{\text{KE}}_{\text{rot}}=\frac{1}{2}{\mathrm{I\omega }}^{2}[/latex]. Both I and ω were found above. Thus, KE rot = (0.5) (4.352 kg·m2)(5.42 rad/s)2 = 64.0 J. The final rotational kinetic energy equals the work done by the torque, which confirms that the work done went into rotational kinetic energy. We could, in fact, have used an expression for energy instead of a kinematic relation to solve part (b). We will do this in later examples. Helicopter pilots are quite familiar with rotational kinetic energy. They know, for example, that a point of no return will be reached if they allow their blades to slow below a critical angular velocity during flight. The blades lose lift, and it is impossible to immediately get the blades spinning fast enough to regain it. Rotational kinetic energy must be supplied to the blades to get them to rotate faster, and enough energy cannot be supplied in time to avoid a crash. Because of weight limitations, helicopter engines are too small to supply both the energy needed for lift and to replenish the rotational kinetic energy of the blades once they have slowed down. The rotational kinetic energy is put into them before takeoff and must not be allowed to drop below this crucial level. One possible way to avoid a crash is to use the gravitational potential energy of the helicopter to replenish the rotational kinetic energy of the blades by losing altitude and aligning the blades so that the helicopter is spun up in the descent. Of course, if the helicopter's altitude is too low, then there is insufficient time for the blade to regain lift before reaching the ground. Determine that energy or work is involved in the rotation. Determine the system of interest. A sketch usually helps. Analyze the situation to determine the types of work and energy involved. For closed systems, mechanical energy is conserved. That is, KEi + PEi = KEf + PEf. Note that KEi and KEf may each include translational and rotational contributions. For open systems, mechanical energy may not be conserved, and other forms of energy (referred to previously as OE), such as heat transfer, may enter or leave the system. Determine what they are, and calculate them as necessary. Eliminate terms wherever possible to simplify the algebra. Check the answer to see if it is reasonable. A typical small rescue helicopter, similar to the one in Figure 5, has four blades, each is 4.00 m long and has a mass of 50.0 kg. The blades can be approximated as thin rods that rotate about one end of an axis perpendicular to their length. The helicopter has a total loaded mass of 1000 kg. (a) Calculate the rotational kinetic energy in the blades when they rotate at 300 rpm. (b) Calculate the translational kinetic energy of the helicopter when it flies at 20.0 m/s, and compare it with the rotational energy in the blades. (c) To what height could the helicopter be raised if all of the rotational kinetic energy could be used to lift it? Strategy Rotational and translational kinetic energies can be calculated from their definitions. The last part of the problem relates to the idea that energy can change form, in this case from rotational kinetic energy to gravitational potential energy. Solution for (a) The rotational kinetic energy is [latex]{\text{KE}}_{\text{rot}}=\frac{1}{2}{\mathrm{I\omega }}^{2}[/latex]. We must convert the angular velocity to radians per second and calculate the moment of inertia before we can find KErot. The angular velocity ω is [latex]\omega =\frac{\text{300 rev}}{\text{1.00 min}}\cdot \frac{\text{2pi rad}}{\text{1 rev}}\cdot \frac{\text{1.00 min}}{\text{60.0 s}}=\text{31.4}\frac{\text{rad}}{\text{s}}[/latex]. The moment of inertia of one blade will be that of a thin rod rotated about its end, found in Figure 5. The total I is four times this moment of inertia, because there are four blades. Thus, [latex]I=4\frac{{\mathrm{M\ell }}^{2}}{3}=4\times \frac{\left(\text{50.0 kg}\right){\left(\text{4.00 m}\right)}^{2}}{3}=\text{1067 kg}\cdot {\text{m}}^{2}[/latex]. Entering ω and I into the expression for rotational kinetic energy gives [latex]\begin{array}{lll}{\text{KE}}_{\text{rot}}& =& 0.5\left(\text{1067 kg}\cdot {\text{m}}^{2}\right){\left(\text{31.4 rad/s}\right)}^{2}\\ & =& 5.26\times {\text{10}}^{5}\text{J}\end{array}[/latex] Solution for (b) Translational kinetic energy was defined in Uniform Circular Motion and Gravitation. Entering the given values of mass and velocity, we obtain [latex]{\text{KE}}_{\text{trans}}=\frac{1}{2}{\mathit{mv}}^{2}=\left(0.5\right)\left(\text{1000 kg}\right){\left(\text{20.0 m/s}\right)}^{2}=2\text{.}\text{00}\times {\text{10}}^{5}\text{J}[/latex]. To compare kinetic energies, we take the ratio of translational kinetic energy to rotational kinetic energy. This ratio is [latex]\frac{2\text{.}text{00}\times {\text{10}}^{5}\text{J}}{5\text{.}\text{26}\times {\text{10}}^{5}\text{J}}=0.380[/latex]. Solution for (c) At the maximum height, all rotational kinetic energy will have been converted to gravitational energy. To find this height, we equate those two energies: KE rot = PE grav or [latex]\frac{1}{2}{\mathrm{I\omega }}^{2}=\text{mgh}[/latex]. We now solve for h and substitute known values into the resulting equation [latex]h=\frac{{\frac{1}{2}\mathrm{I\omega }}^{2}}{\text{mg}}=\frac{5.26\times {10}^{5}\text{J}}{\left(\text{1000 kg}\right)\left(9.80{\text{m/s}}^{2}\right)}=\text{53.7 m}[/latex]. Discussion The ratio of translational energy to rotational kinetic energy is only 0.380. This ratio tells us that most of the kinetic energy of the helicopter is in its spinning blades—something you probably would not suspect. The 53.7 m height to which the helicopter could be raised with the rotational kinetic energy is also impressive, again emphasizing the amount of rotational kinetic energy in the blades. Figure 6. The first image shows how helicopters store large amounts of rotational kinetic energy in their blades. This energy must be put into the blades before takeoff and maintained until the end of the flight. The engines do not have enough power to simultaneously provide lift and put significant rotational energy into the blades. The second image shows a helicopter from the Auckland Westpac Rescue Helicopter Service. Over 50,000 lives have been saved since its operations beginning in 1973. Here, a water rescue operation is shown. (credit: 111 Emergency, Flickr) Conservation of energy includes rotational motion, because rotational kinetic energy is another form of KE . Uniform Circular Motion and Gravitation has a detailed treatment of conservation of energy. How Thick Is the Soup? Or Why Don't All Objects Roll Downhill at the Same Rate? One of the quality controls in a tomato soup factory consists of rolling filled cans down a ramp. If they roll too fast, the soup is too thin. Why should cans of identical size and mass roll down an incline at different rates? And why should the thickest soup roll the slowest? The easiest way to answer these questions is to consider energy. Suppose each can starts down the ramp from rest. Each can starting from rest means each starts with the same gravitational potential energy PEgrav, which is converted entirely to KE, provided each rolls without slipping. KE, however, can take the form of KEtrans or KErot, and total KE is the sum of the two. If a can rolls down a ramp, it puts part of its energy into rotation, leaving less for translation. Thus, the can goes slower than it would if it slid down. Furthermore, the thin soup does not rotate, whereas the thick soup does, because it sticks to the can. The thick soup thus puts more of the can's original gravitational potential energy into rotation than the thin soup, and the can rolls more slowly, as seen in Figure 7. Figure 7. Three cans of soup with identical masses race down an incline. The first can has a low friction coating and does not roll but just slides down the incline. It wins because it converts its entire PE into translational KE. The second and third cans both roll down the incline without slipping. The second can contains thin soup and comes in second because part of its initial PE goes into rotating the can (but not the thin soup). The third can contains thick soup. It comes in third because the soup rotates along with the can, taking even more of the initial PE for rotational KE, leaving less for translational KE. Assuming no losses due to friction, there is only one force doing work—gravity. Therefore the total work done is the change in kinetic energy. As the cans start moving, the potential energy is changing into kinetic energy. Conservation of energy gives PEi = KEf. More specifically, PE grav = KE trans + KE rot or [latex]mgh=\frac{1}{2}{{mv}}^{2}+\frac{1}{2}{{I\omega }}^{2}[/latex]. So, the initial mgh is divided between translational kinetic energy and rotational kinetic energy; and the greater I is, the less energy goes into translation. If the can slides down without friction, then ω = 0 and all the energy goes into translation; thus, the can goes faster. Locate several cans each containing different types of food. First, predict which can will win the race down an inclined plane and explain why. See if your prediction is correct. You could also do this experiment by collecting several empty cylindrical containers of the same size and filling them with different materials such as wet or dry sand. Calculate the final speed of a solid cylinder that rolls down a 2.00-m-high incline. The cylinder starts from rest, has a mass of 0.750 kg, and has a radius of 4.00 cm. Strategy We can solve for the final velocity using conservation of energy, but we must first express rotational quantities in terms of translational quantities to end up with v as the only unknown. Solution Conservation of energy for this situation is written as described above: [latex]mgh=\frac{1}{2}{mv}^{2}+\frac{1}{2}{\mathrm{I\omega }}^{2}[/latex]. Before we can solve for v , we must get an expression for I from Figure 5. Because v and ω are related (note here that the cylinder is rolling without slipping), we must also substitute the relationship ω = v/R into the expression. These substitutions yield [latex]{mgh}=\frac{1}{2}{{mv}}^{2}+\frac{1}{2}\left(\frac{1}{2}{mR}^{2}\right)\left(\frac{{v}^{2}}{{R}^{2}}\right)[/latex]. Interestingly, the cylinder's radius R and mass m cancel, yielding [latex]{gh}=\frac{1}{2}{v}^{2}+\frac{1}{4}{v}^{2}=\frac{3}{4}{v}^{2}[/latex]. Solving algebraically, the equation for the final velocity v gives [latex]v={\left(\frac{4\text{gh}}{3}\right)}^{1/2}[/latex]. Substituting known values into the resulting expression yields [latex]v={\left[\frac{4\left(9.80{\text{m/s}}^{2}\right)\left(\text{2.00 m}\right)}{3}\right]}^{1/2}=\text{5.11 m/s}[/latex]. Discussion Because m and R cancel, the result [latex]v={\left(\frac{4}{3}\text{gh}\right)}^{1/2}[/latex] is valid for any solid cylinder, implying that all solid cylinders will roll down an incline at the same rate independent of their masses and sizes. (Rolling cylinders down inclines is what Galileo actually did to show that objects fall at the same rate independent of mass.) Note that if the cylinder slid without friction down the incline without rolling, then the entire gravitational potential energy would go into translational kinetic energy. Thus, [latex]\frac{1}{2}{mv}^{2}={mgh}[/latex] and [latex]v=(2gh)^{1/2}[/latex], which is 22% greater than [latex](4gh/3)^{1/2}[/latex]. That is, the cylinder would go faster at the bottom. Analogy of Rotational and Translational Kinetic Energy Is rotational kinetic energy completely analogous to translational kinetic energy? What, if any, are their differences? Give an example of each type of kinetic energy. Solution Yes, rotational and translational kinetic energy are exact analogs. They both are the energy of motion involved with the coordinated (non-random) movement of mass relative to some reference frame. The only difference between rotational and translational kinetic energy is that translational is straight line motion while rotational is not. An example of both kinetic and translational kinetic energy is found in a bike tire while being ridden down a bike path. The rotational motion of the tire means it has rotational kinetic energy while the movement of the bike along the path means the tire also has translational kinetic energy. If you were to lift the front wheel of the bike and spin it while the bike is stationary, then the wheel would have only rotational kinetic energy relative to the Earth. Build your own system of heavenly bodies and watch the gravitational ballet. With this orbit simulator, you can set initial positions, velocities, and masses of 2, 3, or 4 bodies, and then see them orbit each other. Click to run the simulation. Section Summary The rotational kinetic energy KErot for an object with a moment of inertia I and an angular velocity ω is given by [latex]{\text{KE}}_{\text{rot}}=\frac{1}{2}{{I\omega }}^{2}[/latex]. Helicopters store large amounts of rotational kinetic energy in their blades. This energy must be put into the blades before takeoff and maintained until the end of the flight. The engines do not have enough power to simultaneously provide lift and put significant rotational energy into the blades. Work and energy in rotational motion are completely analogous to work and energy in translational motion. The equation for the work-energy theorem for rotational motion is, [latex]\text{net }W=\frac{1}{2}{{I\omega }}^{2}-\frac{1}{2}I{{\omega }_{\text{0}}}^{2}[/latex]. 1. Describe the energy transformations involved when a yo-yo is thrown downward and then climbs back up its string to be caught in the user's hand. 2. What energy transformations are involved when a dragster engine is revved, its clutch let out rapidly, its tires spun, and it starts to accelerate forward? Describe the source and transformation of energy at each step. 3. The Earth has more rotational kinetic energy now than did the cloud of gas and dust from which it formed. Where did this energy come from? Figure 8. An immense cloud of rotating gas and dust contracted under the influence of gravity to form the Earth and in the process rotational kinetic energy increased. (credit: NASA) 1. This problem considers energy and work aspects of mass distribution on a merry-go-round (use data from Example 1 as needed. (a) Calculate the rotational kinetic energy in the merry-go-round plus child when they have an angular velocity of 20.0 rpm. (b) Using energy considerations, find the number of revolutions the father will have to push to achieve this angular velocity starting from rest. (c) Again, using energy considerations, calculate the force the father must exert to stop the merry-go-round in two revolutionsx 2. What is the final velocity of a hoop that rolls without slipping down a 5.00-m-high hill, starting from rest? 3. (a) Calculate the rotational kinetic energy of Earth on its axis. (b) What is the rotational kinetic energy of Earth in its orbit around the Sun? 5. A baseball pitcher throws the ball in a motion where there is rotation of the forearm about the elbow joint as well as other movements. If the linear velocity of the ball relative to the elbow joint is 20.0 m/s at a distance of 0.480 m from the joint and the moment of inertia of the forearm is 0.500 kg · m2, what is the rotational kinetic energy of the forearm? 6. While punting a football, a kicker rotates his leg about the hip joint. The moment of inertia of the leg is 3.75 kg · m and its rotational kinetic energy is 175 J. (a) What is the angular velocity of the leg? (b) What is the velocity of tip of the punter's shoe if it is 1.05 m from the hip joint? (c) Explain how the football can be given a velocity greater than the tip of the shoe (necessary for a decent kick distance). 7. A bus contains a 1500 kg flywheel (a disk that has a 0.600 m radius) and has a total mass of 10,000 kg. (a) Calculate the angular velocity the flywheel must have to contain enough energy to take the bus from rest to a speed of 20.0 m/s, assuming 90.0% of the rotational kinetic energy can be transformed into translational energy. (b) How high a hill can the bus climb with this stored energy and still have a speed of 3.00 m/s at the top of the hill? Explicitly show how you follow the steps in the Problem-Solving Strategy for Rotational Energy instructions above. 8. A ball with an initial velocity of 8.00 m/s rolls up a hill without slipping. Treating the ball as a spherical shell, calculate the vertical height it reaches. (b) Repeat the calculation for the same ball if it slides up the hill without rolling. 9. While exercising in a fitness center, a man lies face down on a bench and lifts a weight with one lower leg by contacting the muscles in the back of the upper leg. (a) Find the angular acceleration produced given the mass lifted is 10.0 kg at a distance of 28.0 cm from the knee joint, the moment of inertia of the lower leg is 0.900 kg · m2, the muscle force is 1500 N, and its effective perpendicular lever arm is 3.00 cm. (b) How much work is done if the leg rotates through an angle of 20.0º with a constant force exerted by the muscle? 10. To develop muscle tone, a woman lifts a 2.00-kg weight held in her hand. She uses her biceps muscle to flex the lower arm through an angle of 60.0º. (a) What is the angular acceleration if the weight is 24.0 cm from the elbow joint, her forearm has a moment of inertia of 0.250 kg · m2, and the net force she exerts is 750 N at an effective perpendicular lever arm of 2.00 cm? (b) How much work does she do? 11. Consider two cylinders that start down identical inclines from rest except that one is frictionless. Thus one cylinder rolls without slipping, while the other slides frictionlessly without rolling. They both travel a short distance at the bottom and then start up another incline. (a) Show that they both reach the same height on the other incline, and that this height is equal to their original height. (b) Find the ratio of the time the rolling cylinder takes to reach the height on the second incline to the time the sliding cylinder takes to reach the height on the second incline. (c) Explain why the time for the rolling motion is greater than that for the sliding motion. 12. What is the moment of inertia of an object that rolls without slipping down a 2.00-m-high incline starting from rest, and has a final velocity of 6.00 m/s? Express the moment of inertia as a multiple of MR2, where M is the mass of the object and R is its radius. 13. Suppose a 200-kg motorcycle has two wheels like in Problem 6 from Dynamics of Rotational Motion: Rotational Inertia and is heading toward a hill at a speed of 30.0 m/s. (a) How high can it coast up the hill, if you neglect friction? (b) How much energy is lost to friction if the motorcycle only gains an altitude of 35.0 m before coming to rest? 14. In softball, the pitcher throws with the arm fully extended (straight at the elbow). In a fast pitch the ball leaves the hand with a speed of 139 km/h. (a) Find the rotational kinetic energy of the pitcher's arm given its moment of inertia is 0.720 kg · m2 and the ball leaves the hand at a distance of 0.600 m from the pivot at the shoulder. (b) What force did the muscles exert to cause the arm to rotate if their effective perpendicular lever arm is 4.00 cm and the ball is 0.156 kg? 15. Construct Your Own Problem. Consider the work done by a spinning skater pulling her arms in to increase her rate of spin. Construct a problem in which you calculate the work done with a "force multiplied by distance" calculation and compare it to the skater's increase in kinetic energy. work-energy theorem: if one or more external forces act upon a rigid object, causing its kinetic energy to change from [latex]{\text{KE}}_{\text{1}}[/latex] to [latex]{\text{KE}}_{\text{2}}[/latex] , then the work W done by the net force is equal to the change in kinetic energy rotational kinetic energy: the kinetic energy due to the rotation of an object. This is part of its total kinetic energy 1. (a) 185 J (b) 0.0785 rev (c) W = 9.81 N 3. (a) 2.57 × 1029 (b) [latex]{\text{KE}}_{\text{rot}}=2\text{.}\text{65}\times {\text{10}}^{\text{33}}\text{J}[/latex] 5. KErot = 434 J 7. (a) 128 rad/s (b) 19.9 m 9. (a) 10.4 rad/s2 (b) net W = 6.11 J 14. (a) 1.49 kJ (b) 2.52 × 104 N Rohit Gupta, Brilliant Physics, Matt DeCross, and contributed According to work-kinetic theorem for rotation, the amount of work done by all the torques acting on a rigid body under a fixed axis rotation (pure rotation) equals the change in its rotational kinetic energy: \[{W_\text{torque}} = \Delta K{E_\text{rotation}}.\] Work done by a torque can be calculated by taking an analogy from work done by force. Work done by force is calculated as the dot product of force and displacement of point of application of force. In case of angular motion, force is replaced by torque and linear displacement is replaced by angular displacement. Thus, \[W = \int_{}^{} {\vec \tau \cdot d\vec \theta }. \] Consider a rigid body, rotating freely about a fixed axis of rotation. Its initial angular speed is \({\omega_i}\). Suppose a force \(F\) is now applied (at a distance of \(r\) from the axis of rotation) to increase its angular speed. This force will produce a torque about the axis of rotation: \[\vec \tau = \vec r \times \vec F.\] From the rotational form of Newton's second law, \[{{\vec \tau }_\text{rot}} = {I_\text{rot}}\vec \alpha .\] Here, \({I_\text{rot}}\) is the moment of the body about the axis of rotation and \(\alpha \) is the angular acceleration produced in the body. Also, the work done by torque equals \[W = \int_{}^{} {\tau \cdot d\theta } .\] Angular acceleration changes the angular speed of the body and \[\alpha = \frac{{d\omega }}{{dt}}.\] Therefore, the work done by toque can be written as follows: \[W = \int_{}^{} {{I_\text{rot}}\frac{{d\omega }}{{dt}}\cdot d\theta }. \] Angular velocity is related to the rate of rotation of the body: \[\omega = \frac{{d\theta }}{{dt}}.\] On calculating, one finds \[W = \int_{{\omega _o}}^\omega {I_\text{rot} \omega \, d\omega } = \frac{1}{2}{I_\text{rot}}{\omega ^2} - \frac{1}{2}{I_\text{rot}}\omega _{^o}^2.\] Thus, the work done by the torque equals the change in rotational kinetic energy of the body. Work done by a constant torque is \[W = \tau \theta \] According to the work-kinetic theorem for rotation, work done by torque equals change in rotational kinetic energy \[W = \Delta K{E_\text{rot}}\] \[\tau \theta = \Delta K{E_\text{rot}}\] \[\theta = \frac{{\Delta K{E_\text{rot}}}}{\tau }\] Since, the change in rotational kinetic energy and torque applied are equal, thus the angle rotated by all the objects are same. Skip to main content \(\require{cancel}ewcommand{\N}{\mathbb N} ewcommand{\Z}{\mathbb Z} ewcommand{\Q}{\mathbb Q} ewcommand{\R}{\mathbb R} ewcommand{\lt}{ ewcommand{\amp}{&} \definecolor{fillinmathshade}{gray}{0.9} ewcommand{\fillinmath}[1]{\mathchoice{\colorbox{fillinmathshade}{$\displaystyle $}}{\colorbox{fillinmathshade}{$\textstyle $}}{\colorbox{fillinmathshade}{$\scriptstyle $}}{\colorbox{fillinmathshade}{$\scriptscriptstyle$}}} \) This work can be written in terms of torque and angle \(\Delta \theta\) of rotation. With \(R\) as radius of the wheel, arc distance, \(\Delta s = R\Delta\theta\text{.}\) \begin{equation*} W = F\, \Delta s = F\, R\, \Delta\theta. \end{equation*} The torque by the force applied at the rim is \(\tau = FR\text{.}\) Therefore, the work by the force in rotating the wheel by and angle \(\Delta\theta\) is \begin{equation*} W = \tau \Delta \theta. \end{equation*} This work is called rotational work. (a) How much work is done for each quarter turn? (b) Suppose, you use a \(6\)-in wrench, and apply the same force, except that the force now would act at a distance of about \(5\text{ inches}\) from the center of the bolt. How much work would now be done for each quarter turn? Found typo? In physics, one major player in the linear-force game is work; in equation form, work equals force times distance, or W = Fs. Work has a rotational analog. To relate a linear force acting for a certain distance with the idea of rotational work, you relate force to torque (its angular equivalent) and distance to angle. When force moves an object through a distance, work is done on the object. Similarly, when a torque rotates an object through an angle, work is done. In this example, you work out how much work is done when you rotate a wheel by pulling a string attached to the wheel's outside edge (see the figure). Exerting a force to turn a tire. Work is the amount of force applied to an object multiplied by the distance it's applied. In this case, a force F is applied with the string. Bingo! The string lets you make the handy transition between linear and rotational work. So how much work is done? Use the following equation: W = Fs where s is the distance the person pulling the string applies the force over. In this case, the distance s equals the radius multiplied by the angle through which the wheel turns, so you get right angles to the radius. So you're left with When the string is pulled, applying a constant torque that turns the wheel, the work done equals This makes sense, because linear work is Fs, and to convert to rotational work, you convert from force to torque and from distance to angle. The units here are the standard units for work — joules in the MKS (meter-kilogram-second) system. You have to give the angle in radians for the conversion between linear work and rotational work to come out right. Say that you have a plane that uses propellers, and you want to determine how much work the plane's engine does on a propeller when applying a constant torque of 600 newton-meters over 100 revolutions. You start with the work equation in terms of torque: Plugging the numbers into the equation gives you the work: By the end of this section, you will be able to: Derive the equation for rotational work. Calculate rotational kinetic energy. Demonstrate the Law of Conservation of Energy. In this module, we will learn about work and energy associated with rotational motion. Figure 10.13 shows a worker using an electric grindstone propelled by a motor. Sparks are flying, and noise and vibration are created as layers of steel are pared from the pole. The stone continues to turn even after the motor is turned off, but it is eventually brought to a stop by friction. Clearly, the motor had to work to get the stone spinning. This work went into heat, light, sound, vibration, and considerable rotational kinetic energy. Figure 10.13 The motor works in spinning the grindstone, giving it rotational kinetic energy. That energy is then converted to heat, light, sound, and vibration. (credit: U.S. Navy photo by Mass Communication Specialist Seaman Zachary David Bell) Work must be done to rotate objects such as grindstones or merry-go-rounds. Work was defined in Uniform Circular Motion and Gravitation for translational motion, and we can build on that knowledge when considering work done in rotational motion. The simplest rotational situation is one in which the net force is exerted perpendicular to the radius of a disk (as shown in Figure 10.14) and remains perpendicular as the disk starts to rotate. The force is parallel to the displacement, and so the net work done is the product of the force times the arc length traveled: net W = (net F) Δ s . net W = (net F) Δ s . 10.53 To get torque and other rotational quantities into the equation, we multiply and divide the right-hand side of the equation by rr, and gather terms: net W = (r net F) Δs r . net W = (r net F) Δs r . 10.54 We recognize that r net F= net τr net F= net τ and Δs/r=θΔs/r=θ, so thatnet W=net τθ.net W=net τθ. 10.55 This equation is the expression for rotational work. It is very similar to the familiar definition of translational work as force multiplied by distance. Here, torque is analogous to force, and angle is analogous to distance. The equation net W=net τθnet W=net τθ is valid in general, even though it was derived for a special case.To get an expression for rotational kinetic energy, we must again perform some algebraic manipulations. The first step is to note that net τ=Iαnet τ=Iα, so thatnet W=Iαθ.net W=Iαθ. 10.56 Figure 10.14 The net force on this disk is kept perpendicular to its radius as the force causes the disk to rotate. The net work done is thus net FΔsnet FΔs. The net work goes into rotational kinetic energy. Work and energy in rotational motion are completely analogous to work and energy in translational motion, first presented in Uniform Circular Motion and Gravitation. Now, we solve one of the rotational kinematics equations for αθαθ. We start with the equation ω 2 = ω 0 2 + 2 αθ . ω 2 = ω 0 2 + 2 αθ . 10.57 Next, we solve for αθαθ: αθ = ω 2 − ω 0 2 2 . αθ = ω 2 − ω 0 2 2 . 10.58 Substituting this into the equation for net WW and gathering terms yields net W=12Iω2−12I ω 0 2 .net W=12Iω2−12I ω 0 2 . 10.59 This equation is the work-energy theorem for rotational motion only. As you may recall, net work changes the kinetic energy of a system. Through an analogy with translational motion, we define the term 12Iω212Iω2 to be rotational kinetic energy KErotKErot for an object with a moment of inertia II and an angular velocity ωω: KE rot = 1 2 Iω 2 . KE rot = 1 2 Iω 2 . 10.60 The expression for rotational kinetic energy is exactly analogous to translational kinetic energy, with II being analogous to mm and ωω to vv. Rotational kinetic energy has important effects. Flywheels, for example, can be used to store large amounts of rotational kinetic energy in a vehicle, as seen in Figure 10.15. Figure 10.15 Experimental vehicles, such as this bus, have been constructed in which rotational kinetic energy is stored in a large flywheel. When the bus goes down a hill, its transmission converts its gravitational potential energy into KErotKErot. It can also convert translational kinetic energy, when the bus stops, into KErotKErot. The flywheel's energy can then be used to accelerate, to go up another hill, or to keep the bus from slowing down due to friction. Consider a person who spins a large grindstone by placing her hand on its edge and exerting a force through part of a revolution as shown in Figure 10.16. In this example, we verify that the work done by the torque she exerts equals the change in rotational energy. (a) How much work is done if she exerts a force of 200 N through a rotation of 1.00 rad(57.3º)1.00 rad(57.3º)? The force is kept perpendicular to the grindstone's 0.320-m radius at the point of application, and the effects of friction are negligible. (b) What is the final angular velocity if the grindstone has a mass of 85.0 kg? (c) What is the final rotational kinetic energy? (It should equal the work.)Strategy To find the work, we can use the equation net W=net τθnet W=net τθ. We have enough information to calculate the torque and are given the rotation angle. In the second part, we can find the final angular velocity using one of the kinematic relationships. In the last part, we can calculate the rotational kinetic energy from its expression in KErot=12Iω2KErot=12Iω2.Solution for (a) The net work is expressed in the equation net W=net τθ,net W=net τθ, where net ττ is the applied force multiplied by the radius (rF)(rF) because there is no retarding friction, and the force is perpendicular to rr. The angle θθ is given. Substituting the given values in the equation above yields net W = rF θ= 0.320 m 200 N 1.00 rad = 64.0 N·m. net W = rF θ= 0.320 m 200 N 1.00 rad = 64.0 N·m. Noting that 1 N·m=1 J1 N·m=1 J,net W=64.0 J.net W=64.0 J. Figure 10.16 A large grindstone is given a spin by a person grasping its outer edge. Solution for (b) To find ωω from the given information requires more than one step. We start with the kinematic relationship in the equationω2= ω 0 2 +2αθ.ω2= ω 0 2 +2αθ. Note that ω0=0ω0=0 because we start from rest. Taking the square root of the resulting equation gives Now we need to find αα. One possibility is where the torque is net τ=rF=0.320 m200 N=64.0 N·m.net τ=rF=0.320 m200 N=64.0 N·m. The formula for the moment of inertia for a disk is found in Figure 10.11: I = 1 2 MR 2 = 0.5 85.0 kg 0.320 m 2 = 4.352 kg · m 2 . I = 1 2 MR 2 = 0.5 85.0 kg 0.320 m 2 = 4.352 kg · m 2 . Substituting the values of torque and moment of inertia into the expression for αα, we obtain α=64.0 N·m4.352 kg·m2=14.7rads2.α=64.0 N·m4.352 kg·m2=14.7rads2. Now, substitute this value and the given value for θθ into the above expression for ωω: ω = 2 αθ 1 / 2 = 2 14.7 rad s 2 1.00 rad 1 / 2 = 5.42 rad s . ω = 2 αθ 1 / 2 = 2 14.7 rad s 2 1.00 rad 1 / 2 = 5.42 rad s . Solution for (c) The final rotational kinetic energy is KErot=12Iω2.KErot=12Iω2. Both II and ωω were found above. Thus, KE rot = 0.5 4.352 kg·m25.42 rad/s2=64.0 J. KE rot = 0.5 4.352 kg·m25.42 rad/s2=64.0 J. Discussion The final rotational kinetic energy equals the work done by the torque, which confirms that the work done went into rotational kinetic energy. We could, in fact, have used an expression for energy instead of a kinematic relation to solve part (b). We will do this in later examples. Helicopter pilots are quite familiar with rotational kinetic energy. They know, for example, that a point of no return will be reached if they allow their blades to slow below a critical angular velocity during flight. The blades lose lift, and it is impossible to immediately get the blades spinning fast enough to regain it. Rotational kinetic energy must be supplied to the blades to get them to rotate faster, and enough energy cannot be supplied in time to avoid a crash. Because of weight limitations, helicopter engines are too small to supply both the energy needed for lift and to replenish the rotational kinetic energy of the blades once they have slowed down. The rotational kinetic energy is put into them before takeoff and must not be allowed to drop below this crucial level. One possible way to avoid a crash is to use the gravitational potential energy of the helicopter to replenish the rotational kinetic energy of the blades by losing altitude and aligning the blades so that the helicopter is spun up in the descent. Of course, if the helicopter's altitude is too low, then there is insufficient time for the blade to regain lift before reaching the ground. Determine that energy or work is involved in the rotation. Determine the system of interest. A sketch usually helps. Analyze the situation to determine the types of work and energy involved. For closed systems, mechanical energy is conserved. That is, KEi+PEi=KEf+PEf.KEi+PEi=KEf+PEf. Note that KEiKEi and KEfKEf may each include translational and rotational contributions. For open systems, mechanical energy may not be conserved, and other forms of energy (referred to previously as OEOE), such as heat transfer, may enter or leave the system. Determine what they are, and calculate them as necessary. Eliminate terms wherever possible to simplify the algebra. Check the answer to see if it is reasonable. A typical small rescue helicopter, similar to the one in Figure 10.17, has four blades, each is 4.00 m long and has a mass of 50.0 kg. The blades can be approximated as thin rods that rotate about one end of an axis perpendicular to their length. The helicopter has a total loaded mass of 1000 kg. (a) Calculate the rotational kinetic energy in the blades when they rotate at 300 rpm. (b) Calculate the translational kinetic energy of the helicopter when it flies at 20.0 m/s, and compare it with the rotational energy in the blades. (c) To what height could the helicopter be raised if all of the rotational kinetic energy could be used to lift it?Strategy Rotational and translational kinetic energies can be calculated from their definitions. The last part of the problem relates to the idea that energy can change form, in this case from rotational kinetic energy to gravitational potential energy. Solution for (a) The rotational kinetic energy is KErot=12Iω2.KErot=12Iω2. We must convert the angular velocity to radians per second and calculate the moment of inertia before we can find KErotKErot. The angular velocity ωω isω=300 rev1.00 min·2π rad1 rev·1.00 min60.0 s=31.4rads.ω=300 rev1.00 min·2π rad1 rev·1.00 min60.0 s=31.4rads. The moment of inertia of one blade will be that of a thin rod rotated about its end, found in Figure 10.11. The total II is four times this moment of inertia, because there are four blades. Thus,I=4Mℓ23=4×50.0 kg4.00 m23=1067 kg·m2.I=4Mℓ23=4×50.0 kg4.00 m23=1067 kg·m2. Entering ωω and II into the expression for rotational kinetic energy gives KE rot = 0.5 (1067 kg · m 2) 31.4 rad/s 2 = 5.26 × 10 5 J KE rot = 0.5 (1067 kg · m 2) 31.4 rad/s 2 = 5.26 × 10 5 J Solution for (b) Translational kinetic energy was defined in Uniform Circular Motion and Gravitation. Entering the given values of mass and velocity, we obtainKEtrans=12mv2=0.51000 kg20.0 m/s2=2.00×105 J.KEtrans=12mv2=0.51000 kg20.0 m/s2=2.00×105 J. To compare kinetic energies, we take the ratio of translational kinetic energy to rotational kinetic energy. This ratio is 2.00×105 J5.26×105 J=0.380.2.00×105 J5.26×105 J=0.380. Solution for (c) At the maximum height, all rotational kinetic energy will have been converted to gravitational energy. To find this height, we equate those two energies: KE rot = PE grav KE rot = PE grav or 1 2 Iω 2 = mgh . 1 2 Iω 2 = mgh . We now solve for hh and substitute known values into the resulting equation h = 12 Iω 2 mg = 5.26 × 10 5 J 1000 kg 9.80 m/s 2 = 53.7 m . h = 12 Iω 2 mg = 5.26 × 10 5 J 1000 kg 9.80 m/s 2 = 53.7 m . Discussion The ratio of translational energy to rotational kinetic energy is only 0.380. This ratio tells us that most of the kinetic energy of the helicopter is in its spinning blades—something you probably would not suspect. The 53.7 m height to which the helicopter could be raised with the rotational kinetic energy is also impressive, again emphasizing the amount of rotational kinetic energy in the blades. Figure 10.17 The first image shows how helicopters store large amounts of rotational kinetic energy in their blades. This energy must be put into the blades before takeoff and maintained until the end of the flight. The engines do not have enough power to simultaneously provide lift and put significant rotational energy into the blades. The second image shows a helicopter from the Auckland Westpac Rescue Helicopter Service. Over 50,000 lives have been saved since its operations beginning in 1973. Here, a water rescue operation is shown. (credit: 111 Emergency, Flickr) Conservation of energy includes rotational motion, because rotational kinetic energy is another form of KE KE . Uniform Circular Motion and Gravitation has a detailed treatment of conservation of energy. One of the quality controls in a tomato soup factory consists of rolling filled cans down a ramp. If they roll too fast, the soup is too thin. Why should cans of identical size and mass roll down an incline at different rates? And why should the thickest soup roll the slowest? The easiest way to answer these questions is to consider energy. Suppose each can starts down the ramp from rest. Each can starting from rest means each starts with the same gravitational potential energy PEgravPEgrav, which is converted entirely to KEKE, provided each rolls without slipping. KEKE, however, can take the form of KEtransKEtrans or KErotKErot, and total KEKE is the sum of the two. If a can rolls down a ramp, it puts part of its energy into rotation, leaving less for translation. Thus, the can goes slower than it would if it slid down. Furthermore, the thin soup does not rotate, whereas the thick soup does, because it sticks to the can. The thick soup thus puts more of the can's original gravitational potential energy into rotation than the thin soup, and the can rolls more slowly, as seen in Figure 10.18. Figure 10.18 Three cans of soup with identical masses race down an incline. The first can has a low friction coating and does not roll but just slides down the incline. It wins because it converts its entire PE into translational KE. The second and third cans both roll down the incline without slipping. The second can contains thin soup and comes in second because part of its initial PE goes into rotating the can (but not the thin soup). The third can contains thick soup. It comes in third because the soup rotates along with the can, taking even more of the initial PE for rotational KE, leaving less for translational KE. Assuming no losses due to friction, there is only one force doing work—gravity. Therefore the total work done is the change in kinetic energy. As the cans start moving, the potential energy is changing into kinetic energy. Conservation of energy givesMore specifically, PE grav = KE trans + KE rot PE grav = KE trans + KE rot 10.83 or mgh=12mv2+12Iω2.mgh=12mv2+12Iω2. 10.84 So, the initial mghmgh is divided between translational kinetic energy and rotational kinetic energy; and the greater II is, the less energy goes into translation. If the can slides down without friction, then ω=0ω=0 and all the energy goes into translation; thus, the can goes faster. Locate several cans each containing different types of food. First, predict which can will win the race down an inclined plane and explain why. See if your prediction is correct. You could also do this experiment by collecting several empty cylindrical containers of the same size and filling them with different materials such as wet or dry sand. Calculate the final speed of a solid cylinder that rolls down a 2.00-m-high incline. The cylinder starts from rest, has a mass of 0.750 kg, and has a radius of 4.00 cm.Strategy We can solve for the final velocity using conservation of energy, but we must first express rotational quantities in terms of translational quantities to end up with vv as the only unknown.Solution Conservation of energy for this situation is written as described above: mgh = 1 2 mv 2 + 1 2 Iω 2 . mgh = 1 2 mv 2 + 1 2 Iω 2 . Before we can solve for vv , we must get an expression for II from Figure 10.11. Because vv and ωω are related (note here that the cylinder is rolling without slipping), we must also substitute the relationship ω=v/Rω=v/R into the expression. These substitutions yieldmgh=12mv2+1212mR2v2R2.mgh=12mv2+1212mR2v2R2. Interestingly, the cylinder's radius RR and mass mm cancel, yieldinggh=12v2+14v2=34v2.gh=12v2+14v2=34v2. Solving algebraically, the equation for the final velocity v v givesv=4gh31/2.v=4gh31/2. Substituting known values into the resulting expression yields v=49.80 m/s22.00 m31/2=5.11 m/s.v=49.80 m/s22.00 m31/2=5.11 m/s. Discussion Because mm and RR cancel, the result v=43gh1/2v=43gh1/2 is valid for any solid cylinder, implying that all solid cylinders will roll down an incline at the same rate independent of their masses and sizes. (Rolling cylinders down inclines is what Galileo actually did to show that objects fall at the same rate independent of mass.) Note that if the cylinder slid without friction down the incline without rolling, then the entire gravitational potential energy would go into translational kinetic energy. Thus, 12mv2=mgh12mv2=mgh and v=(2gh)1/2v=(2gh)1/2, which is 22% greater than (4gh/3)1/2(4gh/3)1/2. That is, the cylinder would go faster at the bottom. Analogy of Rotational and Translational Kinetic EnergyIs rotational kinetic energy completely analogous to translational kinetic energy? What, if any, are their differences? Give an example of each type of kinetic energy. Yes, rotational and translational kinetic energy are exact analogs. They both are the energy of motion involved with the coordinated (non-random) movement of mass relative to some reference frame. The only difference between rotational and translational kinetic energy is that translational is straight line motion while rotational is not. An example of both kinetic and translational kinetic energy is found in a bike tire while being ridden down a bike path. The rotational motion of the tire means it has rotational kinetic energy while the movement of the bike along the path means the tire also has translational kinetic energy. If you were to lift the front wheel of the bike and spin it while the bike is stationary, then the wheel would have only rotational kinetic energy relative to the Earth. By the end of this section, you will be able to: Use the work-energy theorem to analyze rotation to find the work done on a system when it is rotated about a fixed axis for a finite angular displacement Solve for the angular velocity of a rotating rigid body using the work-energy theorem Find the power delivered to a rotating rigid body given the applied torque and angular velocity Summarize the rotational variables and equations and relate them to their translational counterparts Thus far in the chapter, we have extensively addressed kinematics and dynamics for rotating rigid bodies around a fixed axis. In this final section, we define work and power within the context of rotation about a fixed axis, which has applications to both physics and engineering. The discussion of work and power makes our treatment of rotational motion almost complete, with the exception of rolling motion and angular momentum, which are discussed in Angular Momentum. We begin this section with a treatment of the work-energy theorem for rotation. Now that we have determined how to calculate kinetic energy for rotating rigid bodies, we can proceed with a discussion of the work done on a rigid body rotating about a fixed axis. Figure shows a rigid body that has rotated through an angle [latex]d\theta[/latex] from A to B while under the influence of a force [latex]\mathbf{\overset{\to }{F}}[/latex]. The external force [latex]\mathbf{\overset{\to }{F}}[/latex] is applied to point P, whose position is [latex]\mathbf{\overset{\to }{r}}[/latex], and the rigid body is constrained to rotate about a fixed axis that is perpendicular to the page and passes through O. The rotational axis is fixed, so the vector [latex]\mathbf{\overset{\to }{r}}[/latex] moves in a circle of radius r, and the vector [latex]d\mathbf{\overset{\to }{s}}[/latex] is perpendicular to [latex]\mathbf{\overset{\to }{r}}.[/latex] Figure 10.39 A rigid body rotates through an angle [latex]d\theta[/latex] from A to B by the action of an external force [latex]\mathbf{\overset{\to }{F}}[/latex] applied to point P. From Figure, we have [latex]\mathbf{\overset{\to }{s}}=\mathbf{\overset{\to }{\theta }}\times \mathbf{\overset{\to }{r}}.[/latex] Thus, [latex]d\mathbf{\overset{\to }{s}}=d(\mathbf{\overset{\to }{\theta }}\times \mathbf{\overset{\to }{r}})=d\mathbf{\overset{\to }{\theta }}\times \mathbf{\overset{\to }{r}}+d\mathbf{\overset{\to }{r}}\times \mathbf{\overset{\to }{\theta }}=d\mathbf{\overset{\to }{\theta }}\times \mathbf{\overset{\to }{r}}.[/latex] Note that [latex]d\mathbf{\overset{\to }{r}}[/latex] is zero because [latex]\mathbf{\overset{\to }{r}}[/latex] is fixed on the rigid body from the origin O to point P. Using the definition of work, we obtain [latex]W=\int \sum \mathbf{\overset{\to }{F}}\cdot d\mathbf{\overset{\to }{s}}=\int \sum \mathbf{\overset{\to }{F}}\cdot (d\mathbf{\overset{\to }{\theta }}\times \mathbf{\overset{\to }{r}})=\int d\mathbf{\overset{\to }{\theta }}\cdot (\mathbf{\overset{\to }{r}}\times \sum \mathbf{\overset{\to }{F}})[/latex] where we used the identity [latex]\mathbf{\overset{\to }{a}}\cdot (\mathbf{\overset{\to }{b}}\times \mathbf{\overset{\to }{c}})=\mathbf{\overset{\to }{b}}\cdot (\mathbf{\overset{\to }{c}}\times \mathbf{\overset{\to }{a}})[/latex]. Noting that [latex](\mathbf{\overset{\to }{r}}\times \sum \mathbf{\overset{\to }{F}})=\sum \mathbf{\overset{\to }{\tau }}[/latex], we arrive at the expression for the rotational work done on a rigid body: [latex]W=\int \sum \mathbf{\overset{\to }{\tau }}\cdot d\mathbf{\overset{\to }{\theta }}.[/latex] The total work done on a rigid body is the sum of the torques integrated over the angle through which the body rotates. The incremental work is [latex]dW=(\sum _{i}{\tau }_{i})d\theta[/latex] where we have taken the dot product in Figure, leaving only torques along the axis of rotation. In a rigid body, all particles rotate through the same angle; thus the work of every external force is equal to the torque times the common incremental angle [latex]d\theta[/latex]. The quantity [latex](\sum _{i}{\tau }_{i})[/latex] is the net torque on the body due to external forces. Similarly, we found the kinetic energy of a rigid body rotating around a fixed axis by summing the kinetic energy of each particle that makes up the rigid body. Since the work-energy theorem [latex]{W}_{i}=\Delta {K}_{i}[/latex] is valid for each particle, it is valid for the sum of the particles and the entire body. The work-energy theorem for a rigid body rotating around a fixed axis is [latex]{W}_{AB}={K}_{B}-{K}_{A}[/latex] where [latex]K=\frac{1}{2}I{\omega }^{2}[/latex] and the rotational work done by a net force rotating a body from point A to point B is [latex]{W}_{AB}=\underset{{\theta }_{A}}{\overset{{\theta }_{B}}{\int }}(\sum _{i}{\tau }_{i})d\theta .[/latex] We give a strategy for using this equation when analyzing rotational motion. Problem-Solving Strategy: Work-Energy Theorem for Rotational Motion Identify the forces on the body and draw a free-body diagram. Calculate the torque for each force. Calculate the work done during the body's rotation by every torque. Apply the work-energy theorem by equating the net work done on the body to the change in rotational kinetic energy. Let's look at two examples and use the work-energy theorem to analyze rotational motion. A [latex]12.0\,\text{N}\cdot \text{m}[/latex] torque is applied to a flywheel that rotates about a fixed axis and has a moment of inertia of [latex]30.0\,\text{kg}\cdot {\text{m}}^{2}[/latex]. If the flywheel is initially at rest, what is its angular velocity after it has turned through eight revolutions? Strategy We apply the work-energy theorem. We know from the problem description what the torque is and the angular displacement of the flywheel. Then we can solve for the final angular velocity. Solution The flywheel turns through eight revolutions, which is [latex]16\pi[/latex] radians. The work done by the torque, which is constant and therefore can come outside the integral in Figure, is [latex]{W}_{AB}=\tau ({\theta }_{B}-{\theta }_{A}).[/latex] We apply the work-energy theorem: [latex]{W}_{AB}=\tau ({\theta }_{B}-{\theta }_{A})=\frac{1}{2}I{\omega }_{B}^{2}-\frac{1}{2}I{\omega }_{A}^{2}.[/latex] With [latex]\tau =12.0\,\text{N}\cdot \text{m},{\theta }_{B}-{\theta }_{A}=16.0\pi \,\text{rad},\,I=30.0\,\text{kg}\cdot {\text{m}}^{2},\,\text{and}\,{\omega }_{A}=0[/latex], we have [latex]12.0\,\text{N-m}(16.0\pi \,\text{rad})=\frac{1}{2}(30.0\,\text{kg}\cdot {\text{m}}^{2})({\omega }_{B}^{2})-0.[/latex] Therefore, [latex]{\omega }_{B}=6.3\,\text{rad}\text{/}\text{s}.[/latex] This is the angular velocity of the flywheel after eight revolutions. Significance The work-energy theorem provides an efficient way to analyze rotational motion, connecting torque with rotational kinetic energy. A string wrapped around the pulley in Figure is pulled with a constant downward force [latex]\mathbf{\overset{\to }{F}}[/latex] of magnitude 50 N. The radius R and moment of inertia I of the pulley are 0.10 m and [latex]2.5\times {10}^{-3}{\text{kg-m}}^{2}[/latex], respectively. If the string does not slip, what is the angular velocity of the pulley after 1.0 m of string has unwound? Assume the pulley starts from rest. Figure 10.40 (a) A string is wrapped around a pulley of radius R. (b) The free-body diagram. Strategy Looking at the free-body diagram, we see that neither [latex]\mathbf{\overset{\to }{B}}[/latex], the force on the bearings of the pulley, nor[latex]M\mathbf{\overset{\to }{g}}[/latex], the weight of the pulley, exerts a torque around the rotational axis, and therefore does no work on the pulley. As the pulley rotates through an angle [latex]\theta ,[/latex] [latex]\mathbf{\overset{\to }{F}}[/latex] acts through a distance d such that [latex]d=R\theta .[/latex] Solution Since the torque due to [latex]\mathbf{\overset{\to }{F}}[/latex] has magnitude [latex]\tau =RF[/latex], we have [latex]W=\tau \theta =(FR)\theta =Fd.[/latex] If the force on the string acts through a distance of 1.0 m, we have, from the work-energy theorem, [latex]\begin{array}{ccc}\hfill {W}_{AB}& =\hfill & {K}_{B}-{K}_{A}\hfill \\ \hfill Fd& =\hfill & \frac{1}{2}I{\omega }^{2}-0\hfill \\ \hfill (50.0\,\text{N})(1.0\,\text{m})& =\hfill & \frac{1}{2}(2.5\times {10}^{-3}{\text{kg-m}}^{2}){\omega }^{2}.\hfill \end{array}[/latex] Solving for [latex]\omega[/latex], we obtain [latex]\omega =200.0\,\text{rad}\text{/}\text{s}.[/latex] Power always comes up in the discussion of applications in engineering and physics. Power for rotational motion is equally as important as power in linear motion and can be derived in a similar way as in linear motion when the force is a constant. The linear power when the force is a constant is [latex]P=\mathbf{\overset{\to }{F}}\cdot \mathbf{\overset{\to }{v}}[/latex]. If the net torque is constant over the angular displacement, Figure simplifies and the net torque can be taken out of the integral. In the following discussion, we assume the net torque is constant. We can apply the definition of power derived in Power to rotational motion. From Work and Kinetic Energy, the instantaneous power (or just power) is defined as the rate of doing work, [latex]P=\frac{dW}{dt}.[/latex] If we have a constant net torque, Figure becomes $W=\tau \theta$ and the power is $P=\frac{dW}{dt}=\frac{d}{dt}(\tau \theta)=\tau \frac{d\theta }{dt}$ or $P=\tau \omega .$ A boat engine operating at $9.0\times {10}^{4}\,\text{W}$ is running at 300 rev/min. What is the torque on the propeller shaft? Strategy We are given the rotation rate in rev/min and the power consumption, so we can easily calculate the torque. Solution $300.0\,\text{rev/min}=31.4\,\text{rad/s;}$ $\tau =\frac{P}{\omega }=\frac{9.0\times {10}^{4}\text{N}\cdot \text{m}\text{/}\text{s}}{31.4\,\text{rad}\text{/}\text{s}}=2864.8\,\text{N}\cdot \text{m}.$ It is important to note the radian is a dimensionless unit because its definition is the ratio of two lengths. It therefore does not appear in the solution. A constant torque of $500\,\text{kN}\cdot \text{m}$ is applied to a wind turbine to keep it rotating at 6 rad/s. What is the power required to keep the turbine rotating? Show Solution 3 MW The rotational quantities and their linear analog are summarized in three tables. Figure summarizes the rotational variables for circular motion about a fixed axis with their linear analogs and the connecting equation, except for the centripetal acceleration, which stands by itself. Figure summarizes the rotational and translational kinematic equations. Figure summarizes the rotational dynamics equations with their linear analogs. Rotational and Translational Variables: Summary Rotational Translational Relationship θ x $\theta =\frac{s}{r}$ ω ${v}_{t}$ $\omega =\frac{{v}_{t}}{r}$ α ${a}_{\text{t}}$ $\alpha =\frac{{a}_{\text{t}}}{r}$ ${a}_{\text{c}}$ ${a}_{\text{c}}=\frac{{v}_{\text{t}}^{2}}{r}$ Rotational and Translational Kinematic Equations: Summary Rotational Translational ${\theta }_{\text{f}}={\theta }_{0}+\overset{-}{\omega }t$ $x={x}_{0}+\overset{-}{v}t$ ${\omega }_{\text{f}}={\omega }_{0}+\alpha t$ ${v}_{\text{f}}={v}_{0}+at$ ${\theta }_{\text{f}}={\theta }_{0}+{\omega }_{0}t+\frac{1}{2}\alpha {t}^{2}$ ${x}_{\text{f}}={x}_{0}+{v}_{0}t+\frac{1}{2}a{t}^{2}$ ${\omega }_{\text{f}}^{2}={\omega }^{2}{}_{0}+2\alpha (\Delta \theta)$ ${v}_{\text{f}}^{2}={v}^{2}{}_{0}+2a(\Delta x)$ Rotational and Translational Equations: Dynamics Rotational Translational $I=\sum _{i}{m}_{i}{r}_{i}^{2}$ m $K=\frac{1}{2}I{\omega }^{2}$ $K=\frac{1}{2}m{v}^{2}$ $\sum _{i}{\tau }_{i}=I\alpha$ $\sum _{i}{\mathbf{\overset{\to }{F}}}_{i}=m\mathbf{\overset{\to }{a}}$ ${W}_{AB}=\underset{{\theta }_{A}}{\overset{{\theta }_{B}}{\int }}(\sum _{i}{\tau }_{i})d\theta$ $W=\int \mathbf{\overset{\to }{F}}\cdot d\mathbf{\overset{\to }{s}}$ $P=\tau \omega$ $P=\mathbf{\overset{\to }{F}}\cdot \mathbf{\overset{\to }{v}}$ A wind turbine rotates at 20 rev/min. If its power output is 2.0 MW, what is the torque produced on the turbine from the wind? Show Solution $\tau =\frac{P}{\omega }=\frac{2.0\times {10}^{6}\text{W}}{2.1\,\text{rad}\text{/}\text{s}}=9.5\times {10}^{5}\text{N}\cdot \text{m}$ A clay cylinder of radius 20 cm on a potter’s wheel spins at a constant rate of 10 rev/s. The potter applies a force of 10 N to the clay with his hands where the coefficient of friction is 0.1 between his hands and the clay. What is the power that the potter has to deliver to the wheel to keep it rotating at this constant rate? A uniform cylindrical grindstone has a mass of 10 kg and a radius of 12 cm. (a) What is the rotational kinetic energy of the grindstone when it is rotating at $1.5\times {10}^{3}\text{rev}\text{/}\text{min}?$ (b) After the grindstone’s motor is turned off, a knife blade is pressed against the outer edge of the grindstone with a perpendicular force of 5.0 N. The coefficient of kinetic friction between the grindstone and the blade is 0.80. Use the work energy theorem to determine how many turns the grindstone makes before it stops. Show Solution a. $K=888.50\,\text{J}$; b. $\Delta \theta =294.6\,\text{rev}$ A uniform disk of mass 500 kg and radius 0.25 m is mounted on frictionless bearings so it can rotate freely around a vertical axis through its center (see the following figure). A cord is wrapped around the rim of the disk and pulled with a force of 10 N. (a) How much work has the force done at the instant the disk has completed three revolutions, starting from rest? (b) Determine the torque due to the force, then calculate the work done by this torque at the instant the disk has completed three revolutions? (c) What is the angular velocity at that instant? (d) What is the power output of the force at that instant? A propeller is accelerated from rest to an angular velocity of 1000 rev/min over a period of 6.0 seconds by a constant torque of $2.0\times {10}^{3}\text{N}\cdot \text{m}$. (a) What is the moment of inertia of the propeller? (b) What power is being provided to the propeller 3.0 s after it starts rotating? Show Solution a. $I=114.6\,\text{kg}\cdot {\text{m}}^{2}$; b. $P=104,700\,\text{W}$ A sphere of mass 1.0 kg and radius 0.5 m is attached to the end of a massless rod of length 3.0 m. The rod rotates about an axis that is at the opposite end of the sphere (see below). The system rotates horizontally about the axis at a constant 400 rev/min. After rotating at this angular speed in a vacuum, air resistance is introduced and provides a force $0.15\,\text{N}$ on the sphere opposite to the direction of motion. What is the power provided by air resistance to the system 100.0 s after air resistance is introduced? A uniform rod of length L and mass M is held vertically with one end resting on the floor as shown below. When the rod is released, it rotates around its lower end until it hits the floor. Assuming the lower end of the rod does not slip, what is the linear velocity of the upper end when it hits the floor? Show Answer $v=L\omega =\sqrt{3Lg}$ An athlete in a gym applies a constant force of 50 N to the pedals of a bicycle to keep the rotation rate of the wheel at 10 rev/s. The length of the pedal arms is 30 cm. What is the power delivered to the bicycle by the athlete? A 2-kg block on a frictionless inclined plane at 40° has a cord attached to a pulley of mass 1 kg and radius 20 cm (see the following figure). (a) What is the acceleration of the block down the plane? (b) What is the work done by the gravitational force to move the block 50 cm? Show Answer a. $a=5.0\,\text{m}\text{/}{\text{s}}^{2}$; b. $W=1.25\,\text{N}\cdot \text{m}$ Small bodies of mass ${m}_{1}\,\text{and}\,{m}_{2}$ are attached to opposite ends of a thin rigid rod of length L and mass M. The rod is mounted so that it is free to rotate in a horizontal plane around a vertical axis (see below). What distance d from ${m}_{1}$ should the rotational axis be so that a minimum amount of work is required to set the rod rotating at an angular velocity $\omega ?$ A cyclist is riding such that the wheels of the bicycle have a rotation rate of 3.0 rev/s. If the cyclist brakes such that the rotation rate of the wheels decrease at a rate of $0.3\,\text{rev}\text{/}{\text{s}}^{2}$, how long does it take for the cyclist to come to a complete stop? Show Solution $\Delta t=10.0\,\text{s}$ Calculate the angular velocity of the orbital motion of Earth around the Sun. A phonograph turntable rotating at 33 1/3 rev/min slows down and stops in 1.0 min. (a) What is the turntable’s angular acceleration assuming it is constant? (b) How many revolutions does the turntable make while stopping? Show Solution a. $0.06\,\text{rad}\text{/}{\text{s}}^{2}$; b. $\theta =105.0\,\text{rad}$ With the aid of a string, a gyroscope is accelerated from rest to 32 rad/s in 0.40 s under a constant angular acceleration. (a) What is its angular acceleration in ${\text{rad/s}}^{2}$? (b) How many revolutions does it go through in the process? Suppose a piece of dust has fallen on a CD. If the spin rate of the CD is 500 rpm, and the piece of dust is 4.3 cm from the center, what is the total distance traveled by the dust in 3 minutes? (Ignore accelerations due to getting the CD rotating.) Show Solution $s=405.26\,\text{m}$ A system of point particles is rotating about a fixed axis at 4 rev/s. The particles are fixed with respect to each other. The masses and distances to the axis of the point particles are ${m}_{1}=0.1\,\text{kg},{r}_{1}=0.2\,\text{m}$, ${m}_{2}=0.05\,\text{kg},{r}_{2}=0.4\,\text{m}$, ${m}_{3}=0.5\,\text{kg},{r}_{3}=0.01\,\text{m}$. (a) What is the moment of inertia of the system? (b) What is the rotational kinetic energy of the system? Calculate the moment of inertia of a skater given the following information. (a) The 60.0-kg skater is approximated as a cylinder that has a 0.110-m radius. (b) The skater with arms extended is approximated by a cylinder that is 52.5 kg, has a 0.110-m radius, and has two 0.900-m-long arms which are 3.75 kg each and extend straight out from the cylinder like rods rotated about their ends. Show Solution a. $I=0.363\,\text{kg}\cdot {\text{m}}^{2}$; b. $I=2.34\,\text{kg}\cdot {\text{m}}^{2}$ A stick of length 1.0 m and mass 6.0 kg is free to rotate about a horizontal axis through the center. Small bodies of masses 4.0 and 2.0 kg are attached to its two ends (see the following figure). The stick is released from the horizontal position. What is the angular velocity of the stick when it swings through the vertical? A pendulum consists of a rod of length 2 m and mass 3 kg with a solid sphere of mass 1 kg and radius 0.3 m attached at one end. The axis of rotation is as shown below. What is the angular velocity of the pendulum at its lowest point if it is released from rest at an angle of $30^\circ?$ Show Answer $\omega =\sqrt{\frac{5.36\,\text{J}}{4.4\,{\text{kgm}}^{2}}}=1.10\,\text{rad}\text{/}\text{s}$ Calculate the torque of the 40-N force around the axis through O and perpendicular to the plane of the page as shown below. Two children push on opposite sides of a door during play. Both push horizontally and perpendicular to the door. One child pushes with a force of 17.5 N at a distance of 0.600 m from the hinges, and the second child pushes at a distance of 0.450 m. What force must the second child exert to keep the door from moving? Assume friction is negligible. Show Solution $F=23.3\,\text{N}$ The force of $20\mathbf{\hat{j}}\text{N}$ is applied at $\mathbf{\overset{\to }{r}}=(4.0\mathbf{\hat{i}}-2.0\mathbf{\hat{j}})\,\text{m}$. What is the torque of this force about the origin? An automobile engine can produce 200 N$\cdot$ m of torque. Calculate the angular acceleration produced if 95.0% of this torque is applied to the drive shaft, axle, and rear wheels of a car, given the following information. The car is suspended so that the wheels can turn freely. Each wheel acts like a 15.0-kg disk that has a 0.180-m radius. The walls of each tire act like a 2.00-kg annular ring that has inside radius of 0.180 m and outside radius of 0.320 m. The tread of each tire acts like a 10.0-kg hoop of radius 0.330 m. The 14.0-kg axle acts like a rod that has a 2.00-cm radius. The 30.0-kg drive shaft acts like a rod that has a 3.20-cm radius. Show Solution $\alpha =\frac{190.0\,\text{N-m}}{2.94\,{\text{kg-m}}^{2}}=64.4\,{\text{rad}\text{/}\text{s}}^{2}$ A grindstone with a mass of 50 kg and radius 0.8 m maintains a constant rotation rate of 4.0 rev/s by a motor while a knife is pressed against the edge with a force of 5.0 N. The coefficient of kinetic friction between the grindstone and the blade is 0.8. What is the power provided by the motor to keep the grindstone at the constant rotation rate? The angular acceleration of a rotating rigid body is given by $\alpha =(2.0-3.0t)\,\text{rad}\text{/}{\text{s}}^{2}$. If the body starts rotating from rest at $t=0$, (a) what is the angular velocity? (b) Angular position? (c) What angle does it rotate through in 10 s? (d) Where does the vector perpendicular to the axis of rotation indicating 0° at $t=0$ lie at $t=10\,\text{s}$? Show Solution a. $\omega =2.0t-1.5{t}^{2}$; b. $\theta ={t}^{2}-0.5{t}^{3}$; c. $\theta =-400.0\,\text{rad}$; d. the vector is at $-0.66(360^\circ)=-237.6^\circ$ Earth’s day has increased by 0.002 s in the last century. If this increase in Earth’s period is constant, how long will it take for Earth to come to rest? A disk of mass m, radius R, and area A has a surface mass density $\sigma =\frac{mr}{AR}$ (see the following figure). What is the moment of inertia of the disk about an axis through the center? Show Answer $I=\frac{2}{5}m{R}^{2}$ Zorch, an archenemy of Rotation Man, decides to slow Earth’s rotation to once per 28.0 h by exerting an opposing force at and parallel to the equator. Rotation Man is not immediately concerned, because he knows Zorch can only exert a force of $4.00\times 1{0}^{7}\text{N}$ (a little greater than a Saturn V rocket’s thrust). How long must Zorch push with this force to accomplish his goal? (This period gives Rotation Man time to devote to other villains.) A cord is wrapped around the rim of a solid cylinder of radius 0.25 m, and a constant force of 40 N is exerted on the cord shown, as shown in the following figure. The cylinder is mounted on frictionless bearings, and its moment of inertia is $6.0\,\text{kg}\cdot {\text{m}}^{2}$. (a) Use the work energy theorem to calculate the angular velocity of the cylinder after 5.0 m of cord have been removed. (b) If the 40-N force is replaced by a 40-N weight, what is the angular velocity of the cylinder after 5.0 m of cord have unwound? Show Answer a. $\omega =8.2\,\text{rad}\text{/}\text{s}$; b. $\omega =8.0\,\text{rad}\text{/}\text{s}$